Analytic Study

of

Harmonic Intervals

by

Dr. Chester D. Mann

Analytic Study of Harmonic Intervals

Published and distributed by:

Pond Science Institute
921 Santa Fe Avenue
La Junta, Colorado 81050
www.svpwiki.com
720-249-2731

Printed in the United States of America

Analytic Study of Harmonic Intervals

4/3 < 7/5 < 3/2

Chester Mann

Larry,

Your dad's book is a nice piece of work. I read about half of it and enjoyed it. Learned a bit, too!

Thanks!

Stanton

5 Sept. 1990

Dear Larry,

This is the book that took so long to write. To console myself, I like to think its value is proportionate to the time and effort I spent on it.

Love, Dad

Analytic Study of Harmonic Intervals

by

Chester D. Mann, Ph.D.

Chester D. Mann
1892 Burnt Mill Road
Tustin, California 92680

1990

This book is a revision and extension of the first three chapters of the author's doctoral dissertation, *A Theory of the Aural Interpretation of Tonal Combinations*, submitted to the Graduate College of the University of Iowa in 1952.

INTRODUCTION

The first step toward understanding the aural interpretation of a musical sound is the determination of the physical nature of the sound itself. To be heard as musical sound, it is essential for vibrations in the air to exhibit at least an element or an approximation of periodicity. The simplest form of periodic vibration is sinusoidal (or simple harmonic) motion, and the sound it makes is called a pure or a simple tone. Most musical tones result from complex forms of vibration that are analyzable, both physically and aurally, in terms of sinusoidal components. Harmonic intervals, being made up of musical tones, are likewise analyzable; and this kind of analysis is regarded here as potentially significant for the theory of harmony.

However, it is known that the ear distorts or transforms the vibrations it receives from the air; and, since sounds are normally heard thru the ear, an analytic study of harmonic intervals must take into consideration the vibrational pattern that the sound assumes in the ear as well as the form it has in the air. The vibrational pattern that an interval assumes in the ear, like that in the air, can be analyzed in terms of sinusoidal components; and these components (or tones) make up what is called here the <u>aural spectrum</u> of an interval. This, then, is the analytic version of an interval that is studied here in an effort to discover principles of importance to the theory of harmony.

The presence or absence in an interval of the beats that result when two or more tones of the spectrum approach unison are considered here as profoundly affecting a musician's judgment of the interval. Only those intervals can be easily and accurately tuned by ear that are determined by the coincidence of spectral tones and, hence, by the elimination of beats. These are called here the discernible intervals, and other intervals are interpreted as approximations, tunings, or mistunings of these.

The aural spectrum of a discernible interval falls according to the frequency ratios of the partials of a complex tone and thereby constitutes a veritable "third tone" that is related in frequency to the two tones of the interval as fundamental to partials. Thus the aural spectrum gives the interval a characteristic sound that is determined by its frequency ratio and, being detected by the musician's ear, plays a major role in the recognition and use of the interval.

The complete set of discernible intervals consists of thirty-two or possibly forty intervals ranging in size from unison to a little less than four octaves and distributed almost evenly within this range so that any interval smaller than four octaves can be said to approximate a discernible interval. The beats that are heard in an approximation of a discernible interval give the approximation a quality of roughness or dissonance, and the discernible interval is said to disturb the approximation. When the difference between consecutive discernible intervals is small enough, these intervals actually disturb each other. In such a case, the more discernible interval is the more disturbing to the less discernible interval; and, as a consequence, the less discernible one is the more dissonant. Mistuned and dissonant intervals are usually avoided or at least used judiciously.

A study of this nature necessarily draws heavily on findings in the fields of acoustics, hearing, and psychology of music as well as theory and history of music. The author has also made extensive use of mathematics, which is now incorporated into the text rather than being relegated to appendixes as it was in his dissertation. The earlier work, being more exploratory in nature and somewhat broader in scope, devoted a chapter to the subject of chords; whereas this book, being more intensive and critical, confines its attention to intervals. As might be expected, then, the approaches and conclusions of the present work, insofar as they differ from those of the earlier writing, are preferred.

Chester D. Mann
Advent 1989

CONTENTS

Contents

Chapter 1

TONES AND INTERVALS

The individual tone is the commonly recognized unit of the musical complex. The reasons for this appear to be partly physical and partly (perhaps to a lesser extent) psychological. The voice and the various musical instruments cannot produce less than a single tone, and the listener intuitively perceives it as an indivisible whole. Yet it is a complex unit with various clearly recognizable attributes. The principal attributes recognized by musicians are heard as being pitch, quality, loudness (or softness), and duration. Quality embraces both timbre and vibrato. Other attributes, such as volume and density, are recognized by some; but they can be related to the principal ones just named.

Most musicians judge the pitch of a tone with ease and precision only in relation to that of another tone. In other words, our appreciation of pitch begins with the comparison of one tone with another. If two tones have the same pitch, we may say that they coincide; if they have different pitches, we say one is higher and the other is lower. Not only can we identify one tone as being higher or lower than another, but we can also identify definite pitch relationships between tones. These pitch relationships are called intervals, but the term <u>interval</u> embraces not only the relationship of the pitches but also the two tones that are sounded to produce the interval. Thus, we distinguish between intervals in which the tones are sounded together and those in which the tones are sounded in succession.

An interval in which the tones are sounded together is called a harmonic interval; one in which the tones occur in succession is known as a melodic interval. This study is concerned with harmonic intervals, in which there is not only a pitch relationship but also a certain sound (or sonority) that is not present in a melodic interval. The author believes that this phenomenon present in harmonic intervals has been a strong shaping force in the development of our polyphonic and homophonic music. As such, it deserves careful analysis.

A. Pitch and Frequency

In making judgments regarding intervals, musicians have been guided principally but not entirely by their sense of hearing. Musical instruments and even specially constructed experimental instruments have contributed to the musician's concepts regarding pitch relationships. Thus, the octave is not defined as merely a certain sensation of pitch relationship but is also considered as a relationship produced by a certain kind of mechanical manipulation of a tone-producing instrument. The ancient Greeks related musical intervals to the ratios of fractional parts of a vibrating string, half a string producing a tone an octave higher than the whole string, two thirds resulting in the fifth above, and so forth.

Galileo (1564-1642) was one of the first authors generally known to relate the size of a musical interval directly to the ratio of the frequencies of the two tones. He stated:

> . . . I assert that the ratio of a musical interval is not immediately determined either by the length, size, or tension of the strings but rather by the ratio of their frequencies, that is, by the number of pulses of air waves which strike the tympanum of the ear, causing it also to vibrate with the same

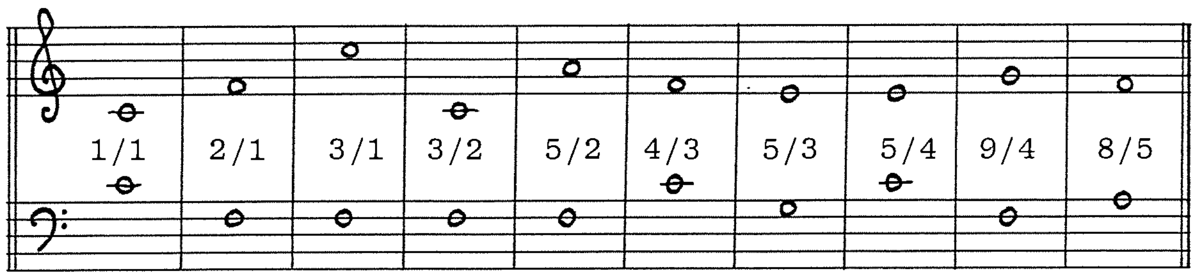


Figure 1. The musical intervals corresponding to certain frequency ratios.

frequency.[1]

He specifically pointed out that the ratio of the frequencies of the octave is 2/1; of the fifth, 3/2; and of the fourth, 4/3. We observe that the ratio of the frequencies of a given interval is the reciprocal of the ratio of the lengths of the fractional parts of a string that will produce the interval. Thus, the fourth higher has 4/3 the frequency but only 3/4 the string length of the tone produced by the whole string. The musical intervals corresponding to certain frequency ratios are shown in Figure 1.

It appears that Galileo must share the honor of the above discovery with Mersenne (1588-1648), who even went so far as to determine frequencies of tones in terms of cycles per second.[2] Thus we find pitch understood to be related to frequency early in the seventeenth century: a greater frequency being heard as a higher pitch; a lesser frequency, as a lower pitch; and equal ratios of frequency being recognized as equal differences in pitch.

---

1. Galileo Galilei, Dialogues Concerning Two New Sciences, p. 103 of the English edition by H. Crew and A. de Salvio. The original edition was published in 1638.

2. Dayton C. Miller, Anecdotal History of the Science of Sound, pp. 13-14. Marin Mersenne published an extended treatise on sound and music, Harmonie Universelle, in 1636.

Recent investigations have shown that under certain conditions pitch is not controlled solely by frequency; other factors that affect the perceived pitch of a tone are intensity, timbre, duration, and inflection.[3] The divergence between pitch and frequency is found to be much greater in many of the psychological studies than it is in musical performance.[4] Experimental procedures which might account for the discrepancies reported include the following: use of pure tones instead of complex tones such as are produced by musical instruments and the human voice, use of intervals and descriptive terminology not recognized in music, exclusive use of melodic rather than harmonic intervals, use of glides instead of steps or leaps, having the subject manipulate apparatus for the production of tone with which he is unfamiliar, use of extreme ranges, and finally the use of nonmusicians and even subjects with poor pitch perception. The elimination of such conditions tends greatly to decrease the extent to which factors other than frequency affect the recognized pitch of a tone.

The term pitch will be used hereafter according to the meaning it has under the conditions of musical experience: complex rather than pure tones, intervals that are recognized or recognizable as possessing distinctive musical qualities, harmonic as well as melodic tonal relationships, discrete instead of continuous pitch changes (at least approximately so), reasonably skilled manipulation of tone production, moderate range, and good pitch perception (a "musical ear"). Under these conditions, the influence of

---

3. Don Lewis, "Pitch: Its Definition and Physical Determinants," University of Iowa Studies in the Psychology of Music, vol. 4 (1937), p. 346f.

4. Stevens and Davis, Hearing, pp. 75-76.

factors other than frequency ratios on pitch relationships is practically negligible.

The present standard of pitch in the United States is 440 cps (cycles per second) for the A in the middle octave of the piano. On this basis, the lowest A of the piano has a frequency of 27.5 cps, coming within approximately a perfect fifth of the lower limit for the perception of pitch; and the highest C has a frequency of 4186 cps in equal temperament, about two octaves below the highest audible frequency.

B. Addition, Subtraction, and Measurement of Intervals

Two intervals may be added by making the lower tone of one coincide with the higher tone of the other; then the sum is the interval formed by the other two tones -- the ones that do not coincide. In other words, the sum of two intervals is the interval composed of the lower tone of one and the higher tone of the other when their other tones coincide. The difference between two intervals is the interval formed by the higher tones of the intervals when their lower tones coincide or by the lower tones of the intervals when their higher tones coincide.

Application of these definitions yields the rules that the frequency ratio of the sum of two intervals equals the *product* of their frequency ratios and that the frequency ratio of the difference between two intervals equals the frequency ratio of the larger interval *divided* by that of the smaller interval. Thus, addition of a major third (with ratio 5/4) and a minor third (with ratio 6/5) results in a perfect fifth, whose frequency ratio (3/2) equals the product of 5/4 and 6/5. Subtraction of a perfect fourth (4/3) from a perfect fifth (3/2) results in a major second whose frequency ratio (9/8) equals the quotient (3/2)/(4/3). Subtraction of a minor third (6/5) from a perfect fourth (4/3) gives another major second (10/9), which is smaller than 9/8.

The difference between these two major seconds is the syntonic comma (81/80). When several intervals of the same name differ by relatively small intervals like this, they are said to differ in tuning or intonation. For example, the major thirds 5/4 and 81/64 differ by a syntonic comma, and 5/4 is said to be in just intonation, whereas 81/64 is in Pythagorean tuning. The addition and subtraction of intervals leads to different tunings for all of the intervals of the diatonic scale. Are all of the tunings of any one interval equally good? If not, which is best, and how bad are the others? Answers to these questions are offered in subsequent pages of this study.

The difference between the pitches of the two tones of an interval is the <u>size</u> of the interval. The addition of two intervals of the same size results in an interval of twice the size, adding an interval to another twice its size results in an interval thrice its size, and so forth. Thus, the relative sizes of the major second, the major third, and the augmented fourth are fittingly expressed by the words tone (whole step), ditone (two steps), and tritone (three steps) that come to us from the Middle Ages.

Let some interval be chosen as a unit interval for the measurement of other intervals, and let its frequency ratio be R. Then the sum of s intervals of unit size is an interval of size s and frequency ratio R^s. If x and y are the respective frequencies of the higher and lower tones of the interval of size s, then its frequency ratio is x/y, and its size and frequency ratio are interrelated precisely by the formula

$$x/y = R^s \qquad \text{B1}$$

whence

$$R = \sqrt[s]{x/y} \qquad \text{B2}$$

and

$$s = \frac{\log (x/y)}{\log R} \qquad \text{B3}$$

Choosing the equally tempered semitone as our unit interval, we find, inasmuch as the octave contains twelve semitones and has the frequency ratio 2/1, that s = 12 when x/y = 2. Substitution of these values into B2 results in

$$R = \sqrt[12]{2} = 1.059463 \qquad \text{B4}$$

The natural logarithm of R, designated by ln R, is .0577623; this and B3 together give us

$$s = 17.31234 \ln (x/y) \qquad \text{B5}$$

as the size in semitones of any given interval.

The cent, which Alexander J. Ellis introduced over a hundred years ago, is one hundredth of a semitone. Both units are used in this study. For example, the size of the pure perfect fifth (x/y = 3/2) is 7.02 semitones or 702 cents. There follows a tabulation of the frequency ratios and sizes of eight curious little intervals that were discovered long ago by taking differences between other, larger intervals:

Name of interval	Freq. ratio	Size in cents
Pythagorean chromatic semitone	2187/2048	113.685
Just diatonic semitone	16/15	111.731
Pythagorean diatonic semitone	256/243	90.225
Just chromatic semitone	25/24	70.672
Diesis (the wolf)	128/125	41.059
Ditonic (or Pythagorean) comma	531441/524288	23.460
Syntonic comma	81/80	21.506
Schisma	32805/32768	1.954

C. Timbre and Partials

Mersenne made also another observation that is important to musical theory. He said:

> . . . every string produces five or more different sounds at the same instant, the strongest of which is called the natural sound of the string, and alone is accustomed to be taken notice of, for the others are so feeble that they are only perceptible by delicate ears. . . Not only the octave and 15th, but also the 12th and major 17th, are always heard; and over and above these I have perceived the major 23rd about the end of the natural sound.[5]

This was about the first intimation that a musical tone has component parts, and that these parts partake of the nature commonly ascribed to the complete tone and stand in fixed intervallic relationships to each other. These parts are now called _partial tones_, or simply _partials_, the one whose frequency corresponds to that of the complete tone being called the first partial or _fundamental_, and the others being called _upper partials_ because thay are higher in pitch. The upper partials of a musical tone are _harmonic_. This means that the second partial (octave higher) has a frequency twice that of the fundamental; the third partial (twelfth higher), three times that of the fundamental; the fourth partial, four times; and so forth. Some of the partials of the tone C are shown in Figure 2 as they are best approximated in the Western musical scale.

Being identical in frequency to the harmonics of a string or pipe, the upper partials are often called _harmonics_, and the sequence of their frequencies is known as the _harmonic series_. A string produces harmonics by vibrating not only as a whole but also in halves, thirds, fourths, fifths, and so forth. This may explain why the series with

---

5. William Pole, _The Philosophy of Music_, p. 41.

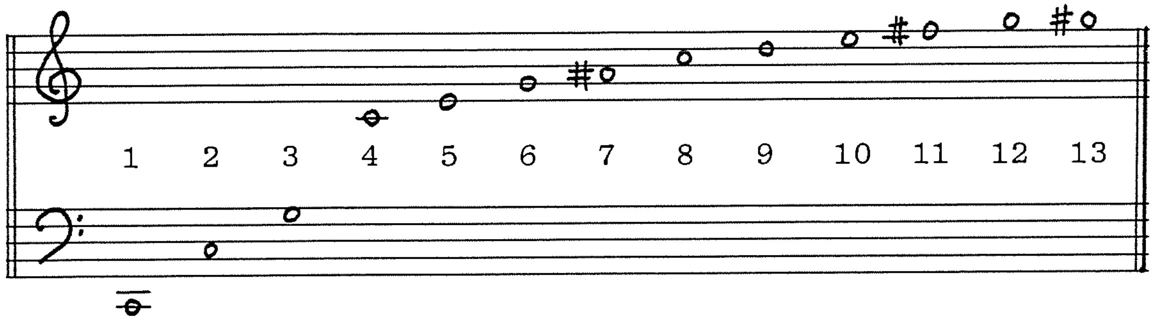


Figure 2. Some of the partials of the tone C as they are best approximated in our musical scale. The numbers show the frequency ratios of the partials to the fundamental and, of course, to each other.

the terms 1, 1/2, 1/3, 1/4, 1/5, . . . is identified in calculus textbooks as the harmonic series. For us, however, the frequency ratios of the harmonics, not their manner of production, are of interest; therefore, any series of frequencies with the relative values 1, 2, 3, 4, 5, . . . is identified here as the harmonic series.

Ohm (1789-1854) formulated in 1843 a more comprehensive and exact notion of this aspect of the musical tone.[6] Miller calls it a "Law of Acoustics" and states it thus:

> This law states that all musical tones are periodic functions; that the ear perceives pendular vibrations alone as simple tones; that all varieties of tone quality are due to particular combinations of a larger or smaller number of simple tones of commensurable frequencies; and that a complex musical tone or a composite mass of musical tones is capable of being analyzed into a sum of simple tones each of which may be separately heard by the ear.[7]

---

6. Georg S. Ohm, "Ueber die Definition des Tones, nebst daran geknüpfter Theorie der Sirene und ähnlicher tonbildender Vorrichtungen," Annalen der Physik und Chemie, Ser. 2, vol. 59 (1843), pp. 513-565.

7. Dayton C. Miller, Anecdotal History of the Science of Sound, pp. 60-61.

A simple tone is the same as a pure tone and cannot be separated into several tones of different frequencies; it presents only one frequency to a listener. The partials of a complex tone are simple tones. The composition of a musical tone as regards the presence and relative intensities of the partials is known as its overtone structure, the term overtone being synonymous with upper partial. A certain aspect of Ohm's law, namely, the relationship of the overtone structure of a tone to its timbre, was the subject of extended experimental investigations by Helmholtz (1821-1894)[8] and Miller.[9] They found that most musical tones could be broken down into component partials with harmonically related frequencies and that timbre was very definitely related to overtone structure.

Ordinarily the partials of a tone are not perceived separately but are rather heard as factors controlling the timbre of a tone whose pitch is identified with that of the fundamental. Nevertheless, it is possible for the ear to pick out the individual partials of a complex tone, especially if a guide tone is sounded for a moment at the frequency of the partial to be heard. Even the elemental nature of a simple tone is not directly apparent to most listeners; they are aware of its being a simple tone through recognition of a timbre or tone quality that has been demonstrated to them as being that of a pure tone.

Table I shows the overtone structures of musical tones produced by a male voice singing ah,[10] and by a violin,[11] a

---

8. Hermann L.F. Helmholtz, Sensations of Tone, Part I.

9. Dayton C. Miller, The Science of Musical Sounds, 1926.

10. After Barrett Stout, "The Harmonic Structure of Vowels in Singing in Relation to Pitch and Intensity," The Journal of the Acoustical Society of America, vol. 10 (1938), pp. 137-146.

Table I

Overtone Structures of Musical Tones

The partials of the tones are listed in the columns beneath the pitches at heights corresponding to the scale of intensity levels at the left. Two or three partials of the same intensity are separated by commas.

Intensity levels in db	Male voice singing ah			Violin			Clarinet			French horn		
	3D	4A	4D	4B♭	5C	6C#	4B♭	5D	6E	2D	4A	5D
			4									
			2						1			
55												
			1									
		3										
		5	3									
50												
								1				1
					1							
	4					1						
45	5											
			12				1					
		2	11									
		1	13									
		4						3	2			
											2	
40	1,2,8											
							3					
		6		2,6	6							
							5					
						4						
35	6	16	10									
	3		9	5	3							
		17		1					3			
								5				
	7,25	13		4	2						1	
	26	7,12	5,6	8		2					3	
30		15					7	2				
	23,24	8,14								5		
	22			3,10								
							9					
	9	18	8									2
	21			7,13	5			4				
							8			6		
25	18,19		7									
	27			9,11					4	7		
		9,10		12		3						
	10	11,19				7	6					
			14		4					4		
					7,13					8		
20	20											
			15				14	6		3		
				19	12		15			9.		3
										2		
	12	20		17			10	8				
	17			13,18	8	6	13	7				
					9		17					
15	13,28											
	14					5	11		5	1		
										10		
	16											
	11,15			14								
	30			15								
10	29				10,11	8						

clarinet,[12] and a French horn.[12] A number before the letter name of a pitch indicates its octave from left to right on the piano keyboard. Each octave starts with A, not C. Thus middle C is designated as 4C, and the major sixth above middle C is 5A. The tones of the voice were roughly characterized as loud; those of the instruments, as rather soft. It can be seen that the relative intensities of the partials are the all-important determinant of the timbres of individual tones. The fundamental is not necessarily the strongest partial; neither does every tone of the same voice or instrument have the same overtone structure. In general, the lower tones have less of their strength in the lower partials, whereas the reverse is true of the higher tones of the same instrument.

That musical tones have upper partials which originate in the manner of vibration of the tone-producing mechanism is well known, but that partials may be heard which do not exist outside the organism of the listener is also true. Helmholtz was probably the first to speak of such. He said:

> Since the human ear easily produces combinational tones, for which the principal causes lying in the construction of that organ have just been assigned, it must also form upper partials for powerful simple tones, as is the case for tuning-forks and the masses of air which they excite in the observations described. Hence we cannot easily have the sensation of a powerful simple tone, without having also the sensation of its harmonic upper partials.[13]

---

11. F.A. Saunders, "The Mechanical Action of Violins," The Journal of the Acoustical Society of America, vol. 9 (1937), pp. 81-98.

12. F.A. Saunders, "Analyses of the Tones of a Few Wind Instruments," The Journal of the Acoustical Society of America, vol. 18 (1946), pp. 395-401.

13. H.L.F. Helmholtz, Sensations of Tone, p. 159.

These "upper partials" that are generated within the ear are now generally known as aural harmonics and, in accord with Helmholtz' explanation, are thought to arise in the nonlinear response of the auditory mechanism.

A similar phenomenon is the fact that the ear hears the pitch of a complex tone as being that of the fundamental even when the fundamental is comparatively weak or missing altogether.[14] This is perhaps best described as a dual phenomenon the two aspects of which present the same frequency to the listener. In the first aspect, a strong enough complex tone without fundamental generates within the ear difference tones of the same frequency as the fundamental. In the second aspect, the frequency (that is, the repetition rate) of this acoustic stimulus actually is that of the fundamental.

D. The Acoustic Spectrum of an Interval

Inasmuch as a harmonic interval consists of two tones sounded together and each tone consists of a number of partials, it follows that a harmonic interval embraces in its physical nature an assemblage of partials standing in various relationships to each other. This assemblage of partials is referred to here as the acoustic spectrum of an interval. Figure 3 represents a portion of the acoustic spectrum of a continuum of intervals from the unison to one that is infinitely large. In this figure, the ordinates measure frequency; the abscissas, points along the continuum, a larger abscissa corresponding to a larger interval. The frequencies of the respective partials of the higher tone are represented by x, 2x, 3x, 4x, and so forth; those

---

14. Stevens and Davis, Hearing, p. 99; C.E. Seashore, Psychology of Music, pp. 68-74; and J.V. Tobias, Foundations of Modern Auditory Theory, Vol. I, Chapter 1, "Periodicity Pitch," by Arnold M. Small.

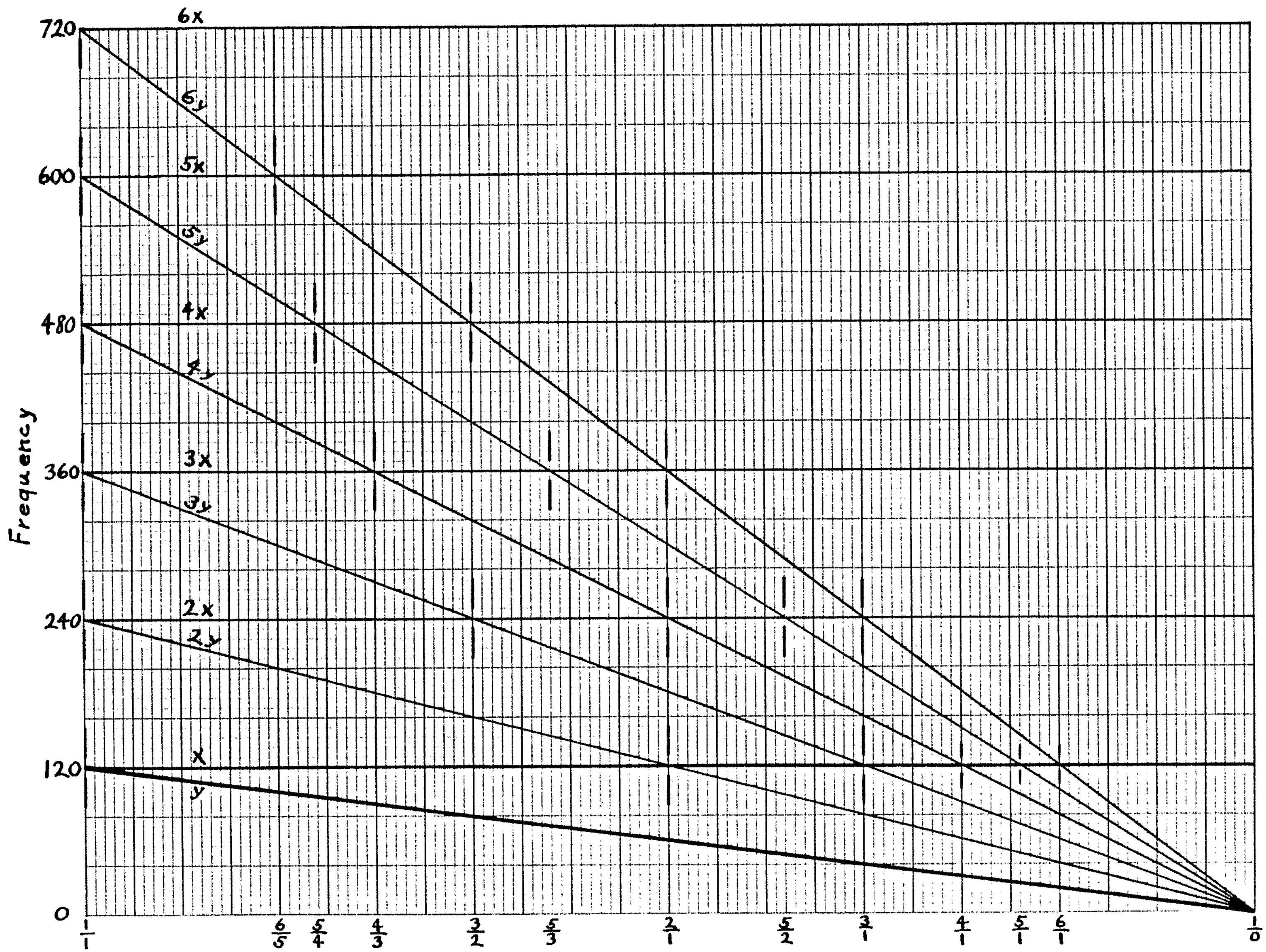


Figure 3. A portion of the acoustic spectrum of a continuum of intervals from the unison to one that is infinitely large.

of the lower tone, by y, 2y, 3y, 4y, and so forth. The higher tone remains at the same frequency throughout, whereas the lower tone is at a lower frequency at each successive point in the continuum, starting at the same frequency as the higher tone and vanishing at the end. By showing only the first six partials in this figure, the writer does not mean to imply that there are none higher; he is rather of the opinion that a practical theory of music will regard every musical tone as if it contained at least six partials.

The spectrum of any particular interval lies along the vertical line that stands at the position of that interval in the continuum. The spectrum of the unison lies along the vertical line at the extreme left of the figure and is composed of the frequencies 120, 240, 360, 480, and so forth; its frequency ratio, 1/1, is marked at the bottom of the figure directly underneath its spectrum; and coinciding partials are pointed out by the vertical dashes above and below each coincidence. The spectrum of the perfect fifth (ratio 3/2) lies along the vertical line that intersects y at the point where y equals 80, and consists of the frequencies 80, 120, 160, 240, 320, 360, 400, 480, and so forth. The coincidence of partials is indicated as before.

The ratios of the frequencies of intervals in which partials are seen to coincide are marked along the axis of abscissas. These intervals are shown in musical notation in Figure 4. Musicians will immediately recognize these as familiar musical intervals in just intonation. The author holds the view that these are commonly recognized largely <u>because</u> of the coincidence of lower partials. If this is so, the inclusion of more partials in Figure 3 would point out more intervals in which partials coincide and therefore are or could be used in music.

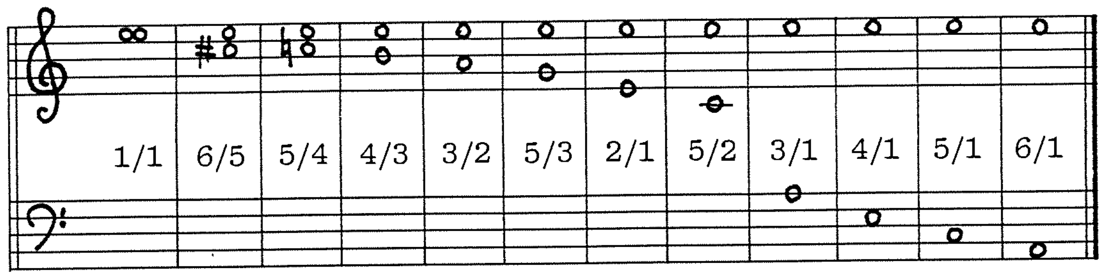


Figure 4. The intervals in which one or more of the first six partials of one tone coincide with one or more of the first six partials of the other tone. Just intonation is required for the exact coincidence of the partials.

Inasmuch as the frequency of a complex tone is that of its fundamental, the frequency ratio of two complex tones is that of their respective fundamentals. It may have been noticed that a definite relationship exists between the frequency ratio of an interval and the ordinal numbers of the partials forming the first (or lowest) coincidence. If the frequency ratio of the higher tone to the lower is 3/2, the lowest coincidence is that of the second partial of the higher tone with the third partial of the lower tone. When the fundamentals have the ratio 5/4, the fourth and fifth partials respectively coincide; and when the fundamentals have the ratio 5/3, the third and fifth partials respectively coincide. These are not the only partials that coincide. In the perfect fourth, not only would the fourth partial of the lower tone coincide with the third partial of the higher but also the eighth partial of the lower would coincide with the sixth of the higher, the twelfth of the lower with the ninth of the higher, and so forth. Out of necessity, the coincidences of the partials always follow an order identical to that of the harmonic series; therefore, it is possible to discover all the coincidences of an interval by determining the lowest coincidence only and remembering that the other coincidences are related to it as the upper partials of a musical tone are to the fundamental.

Let x° and y° be the smallest positive integers that can express the ratio of x to y. Then

$$x/y = x^\circ / y^\circ \qquad \text{D1}$$

and transforming this to

$$y^\circ x = x^\circ y \qquad \text{D2}$$

gives us the perfect expression of the rule we have just presented concerning the coincidence of partials. To understand this, we observe that $y^\circ x$ is the frequency of a partial of the higher tone, $x^\circ y$ is that of a partial of the lower tone, and that the coincidence of partials means equality of their frequencies. D2 gives the <u>lowest</u> coincidence because x° and y° are the <u>smallest</u> whole numbers that can express the ratio of x to y. Higher coincidences can be found simply by multiplying $y^\circ x$ and $x^\circ y$ by the successive integers of the harmonic series.

Intervals in which partials can be found to coincide (even if it is necessary to use very high partials) have frequency ratios that can be expressed in whole numbers. Such ratios are known as <u>commensurable</u> ratios, and intervals with commensurable ratios will be referred to here as commensurable intervals. It is possible for two tones to have a frequency ratio at which no partials can be found to coincide. Such a ratio is called an <u>incommensurable</u> ratio, and intervals with incommensurable ratios will be called incommensurable intervals. Except for the octaves, the equally tempered scale is built entirely in incommensurable intervals.

As in the case of a commensurable interval, the aggregate partials of the acoustic spectrum of an incommensurable interval fall according to a pattern of frequency relationships that is uniquely determined by the ratio of the frequencies of the component tones. This means, on the one hand, that intervals with identical ratios possess a physical likeness to each other and, on the other hand, that intervals with different ratios are physically unlike. From this fact, a reason may be inferred why the frequency ratio of two tones determines the kind and size of the interval, and hence why, in musical practice, equal ratios of frequency are heard as equal differences in pitch.

E. Periodic Motion

The statement (quoted in Section C) that "all musical tones are periodic functions" means that, when the air transmits a musical tone to the ear, the motion of the air in the external auditory canal is a periodic function of time -- which is to say that particle displacement and particle velocity vary with the advance of time in such a way as to be always the same as they were just a period ago. In effect, then, it has been said that the physical essence of musical tone is periodic motion. Figure 5 is a graphic representation of such motion. The point of static equilibrium is understood to be zero displacement. Displacement in one direction from this point is positive displacement; displacement in the opposite direction is negative displacement. Any portion of the path of motion that cannot be divided into an integral number of equal and like parts and yet is continuously repeated in its entirety is called a cycle. The frequency is the number of cycles occurring in one second. The period is the time required for one cycle; it equals the reciprocal of the frequency.

In describing musical tones as periodic functions, we have noted that the repetition of cycles is an essential characteristic, but we have not specified how much repetition is required. One cycle by itself does not constitute periodic motion; two cycles do; but how many cycles are

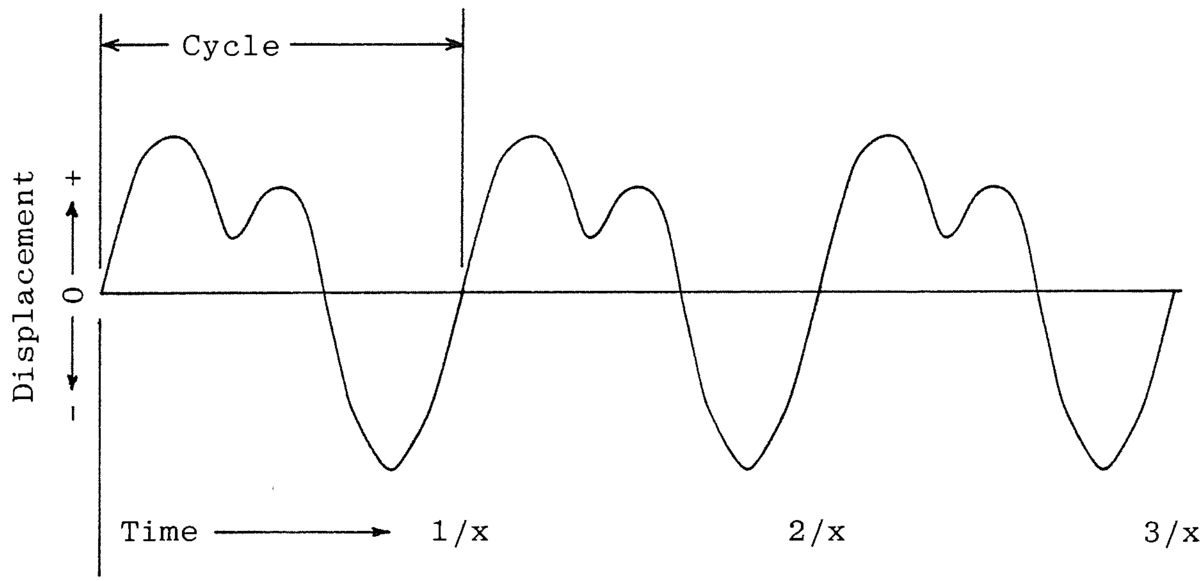


Figure 5. Graphic representation of periodic motion.

needed to convey the sensation of tone? Experimental findings reported by Stevens and Davis (Hearing, pp. 101-102) indicate that at least 3 cycles are required to produce the experience of a definite pitch but that a minimum duration of .011 seconds is also required. The number of cycles or seconds required to perceive the pitch of a tone is a function of frequency. 3 cycles or .06 sec. are required at 50 cps; 7.5 cycles or .015 sec., at 500 cps; 70 cycles or .014 sec., at 5000 cps; more, at higher frequencies. The author estimates that even the shortest durations typical of musical performance satisfy these requirements.

A tone that is strictly periodic is called a steady tone because its pitch, timbre, and loudness are constant. If any or all of these attributes are continuously changing, the tone may be called unsteady. Contrary to the assertion that all musical tones are periodic functions, unsteady tones are common in music. The tone of the piano is unsteady because it is diminishing. A tone sung or played with a vibrato is unsteady because it is pulsating. The ability of the organ and the principal orchestral instruments to produce unsteady as well as steady tones is appreciated by both musicians and nonmusicians. However, the unsteadiness ordinarily encountered in musical practice is sufficiently limited in extent that, while musical tones are not necessarily steady, they are at least approximately so and that, while periodic functions provide accurate mathematical representation of steady tones, they offer approximate representation of unsteady tones. The representation of periodic motion by means of periodic functions follows.

Time will be represented by t; the particle displacement, which varies with t, by u; and the frequency, which is constant, by x. Being periodic, trigonometric functions are used to represent u as a function of t; but they are functions of angles, which we choose to measure in radians; and there are 2π radians per cycle; therefore, the number of radians per second is $2\pi x$. Let

$$\Omega = 2\pi x \qquad \text{E1}$$

and let the particle velocity, which is the derivative of u with respect to t, be represented by v.

T_1, T_2, T_3, and so on are the respective times when the first, second, third, and so on partials contribute zero displacement with maximum velocity to the air in the auditory canal. U_0 is a static displacement; U_1, U_2, U_3, and so on are the respective displacement amplitudes of the first, second, third, and so on partials of a complex tone; and V_1, V_2, V_3, and so on are the respective velocity amplitudes. With these symbols, particle displacement and velocity can be represented as periodic functions of time by means of the two following Fourier series:

$$u = U_0 + U_1 \sin [\Omega(t - T_1)] + U_2 \sin [2\Omega(t - T_2)] + U_3 \sin [3\Omega(t - T_3)] + \ldots \qquad E2$$

$$\begin{aligned} v &= du/dt \\ &= \Omega U_1 \cos [\Omega(t - T_1)] + 2\Omega U_2 \cos [2\Omega(t - T_2)] + 3\Omega U_3 \cos [3\Omega(t - T_3)] + \ldots \\ &= V_1 \cos [\Omega(t - T_1)] + V_2 \cos [2\Omega(t - T_2)] + V_3 \cos [3\Omega(t - T_3)] + \ldots \end{aligned} \qquad E3$$

where

$$V_1 = \Omega U_1,\ V_2 = 2\Omega U_2,\ V_3 = 3\Omega U_3,\ \text{and so on.} \qquad E4$$

The terms $U_1 \sin [\Omega(t - T_1)]$ and $V_1 \cos [\Omega(t - T_1)]$ represent the contribution of the first partial. Two cycles of this contribution are plotted in Figure 6, which shows u versus t as if there were no static displacement and no other partials. This kind of motion consists in oscillations between two points equally distant but opposite in direction from a point of static equilibrium and is known as sinusoidal motion or simple harmonic motion. The displacement amplitude is the distance from the point of equilibrium to one of the extreme points. The kind of tone that this form of vibration gives rise to is called a pure tone or a simple tone.

At the point of greatest positive displacement (A) the particle has no velocity. As it moves from this point to the position of zero displacement (B) its velocity decreases to a minimum ($-V_1$). Continued motion from this point to that of greatest negative displacement (C) is accompanied by increase of the velocity to zero. From this point the direction of motion is reversed, and the velocity increases, reaching a maximum (V_1) at the point of zero displacement (D). B and D are identical positions of the particle, the difference being that D is passed half a period later than B

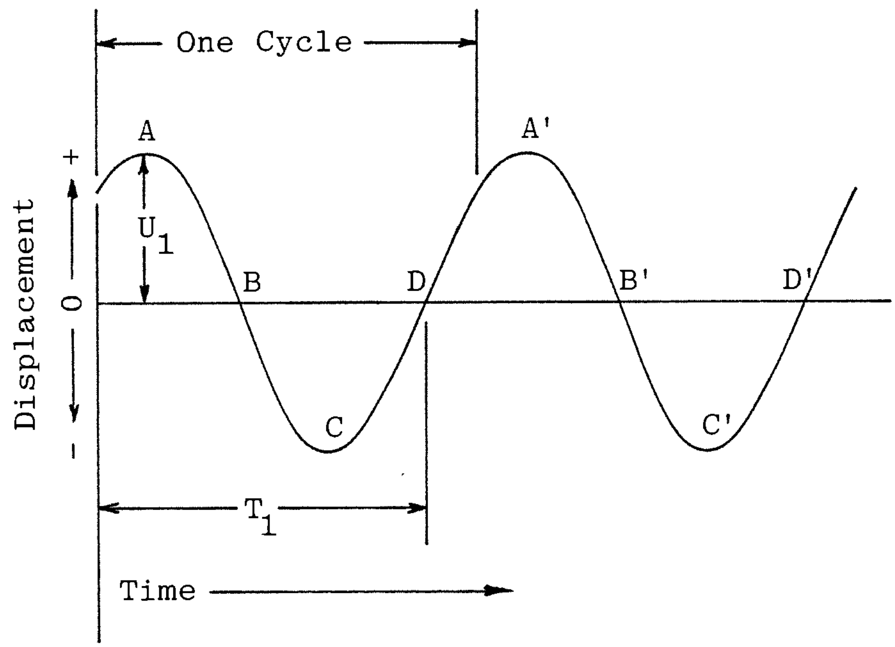


Figure 6. Sinusoidal motion. U_1 is the displacement amplitude. A, B, C, D, and so forth point out the path of the motion with respect to time. T_1 is the time at which the particle first passes from negative to positive displacement.

and in the opposite direction. Continued motion from the point of equilibrium (D) returns the particle to the position of maximum positive displacement (A') and is accompanied by reduction of the velocity to zero once again. This completes the path of motion; continued motion merely repeats the path just described. A' is the same position as A, the distinction being that it is reached a full period later; similar distinctions exist between B and B', C and C', and D and D'.

It is not of interest here to discuss the ways in which the voice and the musical instruments produce a complex form of periodic vibration such as is shown in Figure 5, but it is of interest to understand that such a form of vibration can be resolved into sinusoidal components such as the terms of E2 and E3. Discovering the sinusoidal components of a complex periodic function is known as harmonic analysis and can be accomplished with the aid of Fourier's Theorem, which has been presented in many mathematical works. J.B.J. Fourier presented the mathematical procedure for the harmonic analysis of periodic vibrations in 1822 in his La Théorie Analytique de la Chaleur.

F. Loudness and Intensity

Loudness has been defined as "the magnitude of an auditory sensation."[15] Obviously the loudness of a tone of constant frequency and overtone structure will correspond roughly to the amplitude of the vibration. It is also true that, for a tone of constant displacement amplitude and composition, the loudness is greater for a higher frequency and less for a lower frequency. The one physical property of sound that is found to be principally responsible for the experience of loudness is called intensity or sound intensity and, for sound waves in the air, can be defined in terms of the rate of flow of energy through a unit area of the medium.

In this kind of energy flow, certain particles of air are displaced by the pressure of other particles, and work is done. This pressure is an alteration from the atmospheric pressure; it is called excess air pressure and is represented by p. Let the density of the air at atmospheric pressure be represented by D, and let the speed of sound in air be signified by c. Then the excess pressure due to sound is related to the particle velocity in the following simple way:

---

15. Fletcher and Munson, "Loudness, Its Definition, Measurement and Calculation," The Journal of the Acoustical Society of America, vol. V (1933), pp. 82-108.

$$p = Dcv \tag{F1}$$

If substitution is made for v from E3 into F1, it is seen that p varies periodically in just the same manner as v. This periodic alteration in pressure on the ear drum activates the mechanism of the ear so as to lead to the sensation of tone.

Let the work done be represented by W, and let us apply the formula: work equals the force times the distance thru which it acts. Recognizing that the force (p in this case) is a function of time, we have recourse to differentials and write

$$dW = pdu \tag{F2}$$

Since $v = du/dt$ by definition,

$$du = vdt \tag{F3}$$

and substitution from this into F2 results in

$$dW = pvdt \tag{F4}$$

Integrating this thru one period gives

$$W_o = \int_0^{1/x} pvdt \tag{F5}$$

which is the work done in one cycle.

The intensity, defined as the rate of flow of energy and denoted by I, is the work done per second; therefore, it is obtained by multiplying the work done in one cycle by the number of cycles per second, with the result

$$\begin{aligned} I &= xW_o \\ &= x\int_0^{1/x} pvdt \end{aligned} \tag{F6}$$

Substitution into this from F1 gives

$$\begin{aligned} I &= x\int_0^{1/x} Dcv^2dt \\ &= Dcx\int_0^{1/x} v^2dt \end{aligned} \tag{F7}$$

Using the expression given for v in E3 and performing the indicated integration gives us

$$\int_0^{1/x} v^2 dt = \frac{1}{2x}(V_1^2 + V_2^2 + V_3^2 + \ldots) \qquad \text{F8}$$

whence, because of F7,

$$I = \tfrac{1}{2} Dc(V_1^2 + V_2^2 + V_3^2 + \ldots)$$

$$= I_1 + I_2 + I_3 + \ldots \qquad \text{F9}$$

where

$$I_1 = \tfrac{1}{2} DcV_1^2, \quad I_2 = \tfrac{1}{2} DcV_2^2, \quad I_3 = \tfrac{1}{2} DcV_3^2, \quad \ldots \qquad \text{F10}$$

It is seen that the intensity of a complex tone is the sum of the intensities of its partials.

It has been found that equal differences of loudness of a tone of constant frequency correspond better to equal ratios of intensity (Weber-Fechner law) than to equal differences of intensity. A scale in which equal ratios are represented numerically as equal differences is one in which each value is represented by its logarithm -- the power to which a certain number, called the base, must be raised to produce the given value. In such a scale the zero point does not represent zero intensity but rather an arbitrarily chosen value. One recommended value is 10^{-10} microwatts per square centimeter; it has the advantage of falling reasonably near the threshold of audibility at a frequency of 1000 cycles per second.

If the zero point or "reference-intensity" is represented by I_o, the "intensity-level" (IL) in decibels (db) can be formulated thus:

$$IL = 10 \log_{10} I - 10 \log_{10} I_o \qquad \text{F11}$$

With microwatts per square centimeter as the unit of intensity and $I_o = 10^{-10}$ microwatts per square centimeter,

$$IL = 10 \log_{10} I + 100 \qquad \text{F12}$$

The intensity range of the ear at the frequency at which the ear is most sensitive is about 140 decibels. The intensity level of ordinary conversation is around 60 decibels; that of musical performance may range from about 25 to 95 decibels.

The loudness of a tone, as judged by the ear, does not always correspond exactly to its intensity level. For high intensity levels and over a considerable range of frequencies there is near correspondence, but for low intensities and extremely low or high frequencies there is a great difference. A sound of low frequency (below 100 cps) or of high frequency (above 4000 cps) must be more intense than a sound of medium frequency to equal it in loudness. The perceived loudness of a tone is also affected by its duration, greater durations being required at lower frequencies for equal loudnesses, thereby suggesting that the absolute number of cycles conditions the sensed loudness of a tone. See Wever, Theory of Hearing, pp. 317-319.

G. Phase Relations in Complex Tones

In periodic motion, a phase is any point or stage in a cycle of the motion. In Figure 6, the points A, B, C, and D are phases of sinusoidal motion. Here, the motion is given by

$$u = U_1 \sin [\Omega(t - T_1)] \qquad \text{G1}$$

$$v = V_1 \cos [\Omega(t - T_1)] \qquad \text{G2}$$

and phase can be identified by knowledge of the angle $\Omega(t - T_1)$, whose values at A, B, C, and D are respectively $-3\pi/2$, $-\pi$, $-\pi/2$, and 0.

G1 and G2 contain only the contribution of the first partial. In order to facilitate inclusion and discussion of all the partials, let n equal the number of partials, and let

$$r = 1, 2, 3, \ldots n \qquad \text{G3}$$

Then $U_r \sin [r\Omega(t - T_r)]$ and $V_r \cos [r\Omega(t - T_r)]$ can represent the contribution of any partial according to the value of r, and the summation sign $\sum$ can be used to unite the contributions of all the partials in expressions equivalent to E2 and E3 as follows:

$$u = U_0 + \sum_{r=1}^{n} U_r \sin [r\Omega(t - T_r)] \qquad \text{G4}$$

$$v = \sum_{r=1}^{n} V_r \cos [r\Omega(t - T_r)] \qquad \text{G5}$$

Let l_r be zero or a positive or negative integer or integral variable, and let

$$\begin{aligned} P_r &= r\Omega(t - T_r) - 2\pi l_r \\ &= 2\pi[rx(t - T_r) - l_r] \end{aligned} \qquad \text{G6}$$

Then

$$\sin P_r = \sin [r\Omega(t - T_r)] \qquad \text{G7}$$

and

$$\cos P_r = \cos [r\Omega(t - T_r)] \qquad \text{G8}$$

regardless of the particular integral value of l_r at any time; and $r\Omega(t - T_r)$ in G4 and G5 can be replaced by P_r:

$$u = U_0 + \sum_{r=1}^{n} U_r \sin P_r \qquad \text{G9}$$

$$v = \sum_{r=1}^{n} V_r \cos P_r \qquad \text{G10}$$

In view of this and the definition of T_r in Section E, we see that T_r can also be defined as a time (or the first time) when P_r equals 0 or a multiple of 2π.

P_r is called the phase angle. If l_r is restricted by the condition that

$$0 \leqq rx(t - T_r) - l_r < 1 \qquad \text{G11}$$

then

$$0 \leqq P_r < 2\pi \qquad \text{G12}$$

and P_r becomes a periodic function of the time with frequency rx. As such, it offers us an explicit identification of the phase. This is seen in the following tabulation, which applies to Figure 6:

Phase	A	B	C	D	A'	B'	C'	D'
$\Omega(t - T_1)$	$-3\pi/2$	$-\pi$	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π
P_1	$\pi/2$	π	$3\pi/2$	0	$\pi/2$	π	$3\pi/2$	0

As defined in G3, r can designate any partial. In order to be able to designate any other partial in the same way, we introduce s and, like r, let

$$s = 1, 2, 3, \ldots n \qquad \text{G13}$$

Then, following the example of G6,

$$P_s = 2\pi[sx(t - T_s) - l_s] \qquad \text{G14}$$

and we may now look into phase relations in complex tones, that is, phase relations between different partials of the same complex tone.

When two pure tones have the same frequency, the difference between their phase angles is constant; and this "phase difference" is sufficient to characterize the phase relationship between the two tones. In a complex tone, no two partials have the same frequency; and the phase difference, obtained by subtracting G6 from G14, is

$$P_s - P_r = 2\pi[x(s - r)t - x(sT_s - rT_r) - l_s + l_r] \qquad \text{G15}$$
$$= \text{a variable.}$$

This is not an apt characterization of the phase relationship between two partials.

It is desirable to find, if possible, a phase relationship that does not change with the advance of time. This is accomplished by the elimination of t between G6 and G14, with the result that, at all times,

$$rP_s - sP_r = 2\pi[rsx(T_r - T_s) + sl_r - rl_s] \qquad \text{G16}$$
$$= \text{a constant.}$$

$sl_r - rl_s$ is a multiple of the greatest common divisor of r and s. If r and s are relatively prime, that is, if their greatest common divisor is 1, then $sl_r - rl_s$ is an integer and may be chosen so that

$$0 \leqq rsx(T_r - T_s) + sl_r - rl_s < 1$$
$$0 \leqq rP_s - sP_r < 2\pi \qquad \text{G17}$$

G16 offers us a clear and concise characterization of the phase relationship between any two partials, such that different constant values of $rP_s - sP_r$ represent different phase relationships. To identify the phase relations among all the partials of one tone, probably the simplest and most systematic procedure is to relate the phase angles of all the upper partials to that of the fundamental. To this end, we set r = 1 to designate the fundamental, and let s > 1 to represent any upper partial:

$$P_s - sP_1 = 2\pi[sx(T_1 - T_s) + sl_1 - l_s] \qquad \text{G18}$$

$sl_1 - l_s$ is an integer. If it is chosen so that

$$0 \leqq sx(T_1 - T_s) + sl_1 - l_s < 1 \qquad \text{G19}$$

then

$$0 \leqq P_s - sP_1 < 2\pi \qquad \text{G20}$$

and, when $P_1 = 0$, P_s is subject to the same limits as P_r is in G12.

Figures 7 and 8 present P_1 and P_2 as functions of t in

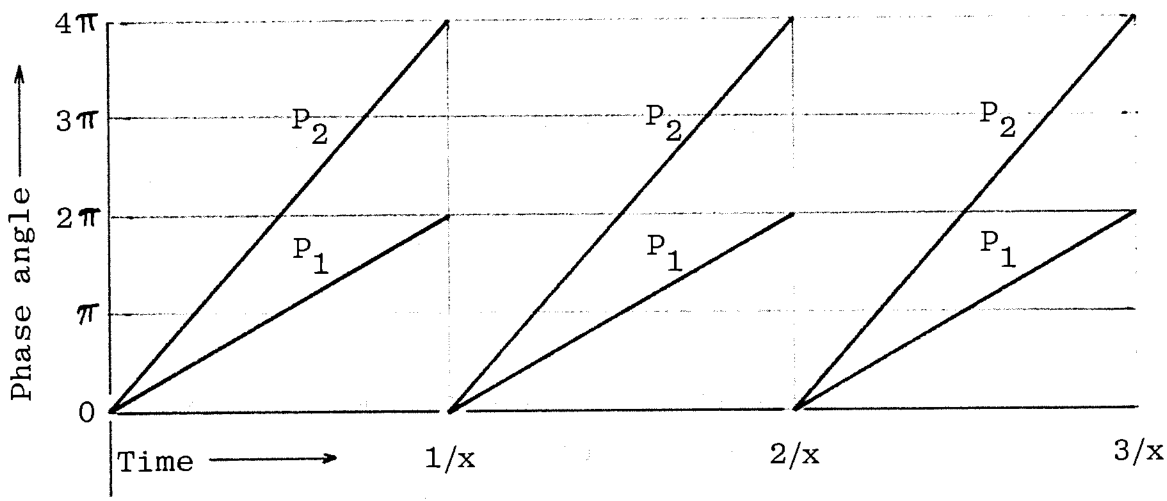


Figure 7. Phase relation $P_2 - 2P_1 = 0$. $T_1 = 0$. $T_2 = 0$.

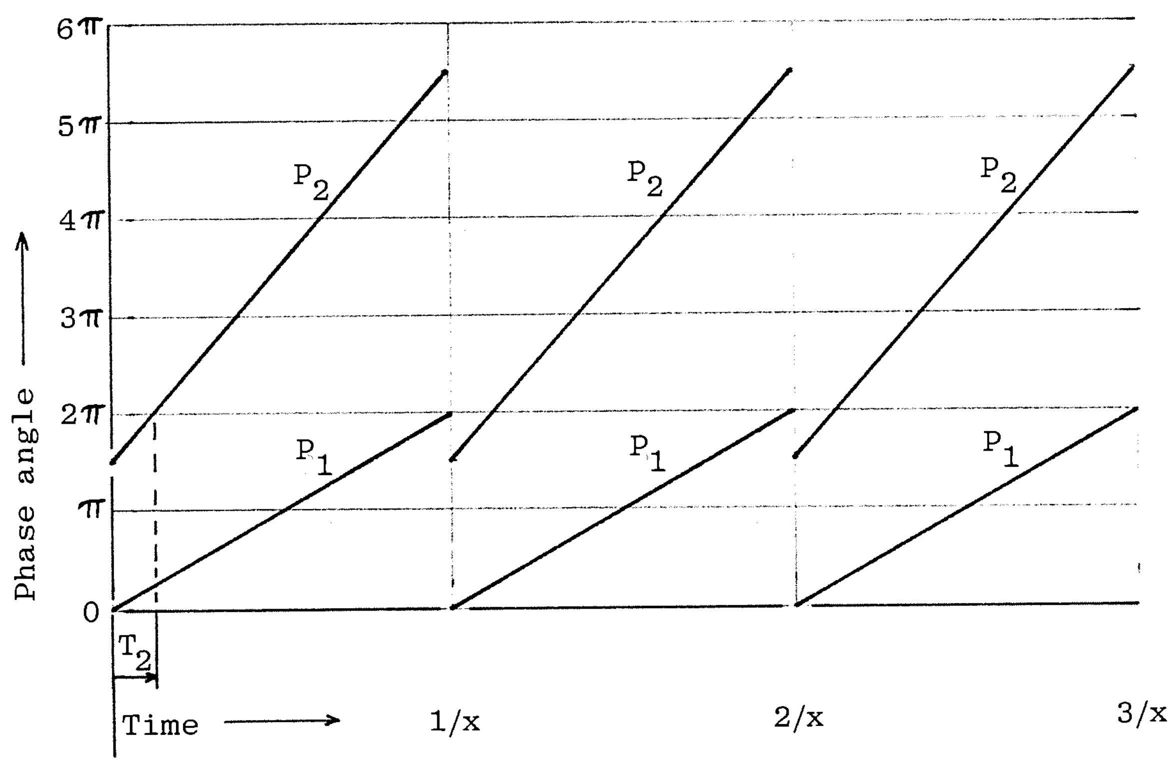


Figure 8. Phase relation $P_2 - 2P_1 = 3\pi/2$. $T_1 = 0$.
$T_2 = 1/(8x)$.

two of the many possible phase relations. Employing these in G9 with n = 2 gives us the equations for the corresponding paths of motion of the air in the auditory canal. In Figure 7, $P_2 = 2P_1$; and

$$u = U_0 + U_1 \sin P_1 + U_2 \sin 2P_1 \qquad \text{G21}$$

In Figure 8, $P_2 = 2P_1 + 3\pi/2$; and

$$u = U_0 + U_1 \sin P_1 + U_2 \sin (2P_1 + 3\pi/2) \qquad \text{G22}$$

Curves C and D in Figure 31 of Helmholtz' Sensations of Tone show the visibly different vibration patterns that follow from G21 and G22.

Helmholtz was concerned as to whether different phase relations, hence different paths of motion such as these, make the same or different impressions on the ear. In question is whether or not phase relations in complex tones affect the quality of tone. If changes in phase relations between the partials of complex tones make no difference in the way the tones sound to the ear, then tone quality is not affected. If changes in phase relations make different impressions on the ear, then the quality of a musical tone is affected.

After having experimented upon "numerous combinations of tone with varied differences of phase" without experiencing "the slightest difference in the quality of tone," Helmholtz laid it down as a law that

> differences in musical quality of tone depend solely on the presence and strength of partial tones, and in no respect on the differences in phase under which these partial tones enter into composition.[16]

We have seen that differing phase relations in complex tones give rise to different vibration patterns. Now different vibration patterns present physically different

---

16. Sensations of Tone, p. 127.

stimuli to the ear, yet Helmholtz found that these different stimuli resulting from phase differences evoke the same auditory sensation. Some investigators have been understandably reluctant to accept such a finding as a law, or they have accepted it only with reservations. Let us briefly consider what three of these have said.

During Helmholtz' lifetime, R. Koenig wrote:

> Hence, although the quality of tone principally depends on the number and relative intensity of the harmonic tones compounded, the influence of difference of phase is not by any means so insignificant as to be entirely negligible. We may say, in general terms, that the differences in the number and relative intensity of the harmonic tones compounded produces those differences in the quality of tone which are remarked in musical instruments of different families, or in the human voice uttering different vowels. But the alteration of phase between these harmonic tones can excite at least such differences of quality of tone as are observed in musical instruments of the same family, or in different voices singing the same vowel.[17]

In 1938, C.E. Seashore published the following statement:

> Timbre is that characteristic of a tone which depends upon its harmonic structure as modified by absolute pitch and total intensity. The harmonic structure is expressed in terms of the number, distribution, and relative intensity of its partials. Recent experiments show that we must also take phase relations into account. Physically the timbre of the tone is a cross section of the tone quality for the moment represented by the duration of one vibration in the sound.[18]

More recently, J. Roederer wrote as follows:

> Psychoacoustical experiments with electronically generated steady complex tones, of equal pitch and loudness but different spectra and phase relation-

---

17. Translator's appendix to Sensations of Tone, p. 537.

18. Psychology of Music, p. 97.

> ships among the harmonics, show that the timbre sensation is controlled primarily by the power spectrum (Plomp 1970). Phase changes, although clearly perceptible, particularly when effected among the high frequency components, play only a secondary role.[19]

The power spectrum referred to here is the same thing as the overtone structure, defined in Section C, and the harmonic structure, referred to by Seashore. While these three authors question Helmholtz' conclusion, they nevertheless show general agreement that timbre depends primarily on overtone structure and only secondarily on phase relations.

H. Vibrato: Periodic Pulsation

There can be no doubt that the vibrato has been used for centuries; one might speculate as to whether or not it is as old as music itself. Practically all singers use it; in fact, it is difficult to produce a good vocal tone without a vibrato. The clavichord, which is capable of a pleasing vibrato, called in German _Bebung_, dates back to the 14th century. Martin Agricola, in his _Musica instrumentalis deudsch_ (ed. 1545), refers to Polish fiddlers:

> Who, while their stopping fingers teeter,
> Produce a melody much sweeter
> Than 'tis on other fiddles done.[20]

The scientific study of the vibrato is of recent origin, however. The first published experimental investigation of the subject was by Schoen in 1922.[21] A review of this

---

19. Juan G. Roederer, _Introduction to the Physics and Psychophysics of Music_, second edition, p. 136.

20. Curt Sachs, _Our Musical Heritage_ (1948), p. 160. Agricola's work was first issued in 1528.

21. Max Schoen, "An Experimental Study of the Pitch Factor in Artistic Singing," _Psychol. Monogr._, vol. 31 (1922), pp. 230-259.

publication is given in that author's more recent book, The Psychology of Music, pp. 202-205, according to which he found that the vocal vibrato consisted of synchronous pulsations of both frequency and intensity limited as to extent of frequency variation[22] and rate of pulsation; that the listener appreciated the vibrato principally as a factor of tone quality; and that "the vibrato was a basic, fundamental attribute of an effective singing voice." He also pointed out that the listener can hardly distinguish between a frequency oscillation and an intensity oscillation, "especially when the fluctuations take place rapidly and periodically."

In 1932 the University of Iowa published the first volume of Studies in the Psychology of Music under the editorship of Carl E. Seashore. This volume was devoted entirely to the subject of the vibrato and contained material gathered in experimental studies made at the University of Iowa. Accurate techniques had been developed for determining certain facts about the physical nature of vibratos produced by performers, and apparatus had been constructed to produce synthetic vibratos in which the various factors were under control.

Some of the physical facts brought to light were that the vibrato is very common in the voices of singers and in the tones of the bowed string instruments; that periodic variations in the overtone structure constitute a third factor in the nature of the vibrato; that the variation of frequency with time is approximately sinusoidal; that the average extent of frequency variation in the vibrato of the violin, viola, and cello is approximately a quarter tone; that the average extent of the vocal vibrato is approximately a

---

22. The extent is measured from the lowest to the highest frequency reached in the fluctuation.

semitone; that the average rate of pulsation of both kinds of vibrato is between six and seven per second, except that the rate for the cello is somewhat slower; that the average extent of intensity fluctuation is about 4.2 decibels for string instruments and about 2.4 db for voices; that the fluctuations of frequency, intensity, and overtone structure are synchronous; and that the frequency factor is more important than the intensity factor both physically and psychologically.

The facts as to how the vibrato is heard are at least equally important here. It was found that the vibrato is not always appreciated as such, being heard rather as a factor in tone quality, especially with untrained observers. Trained observers could distinguish easily between the contributions of vibrato and of overtone structure to tone quality. This led to the adoption of the term _sonance_ (successive fusion) to designate the contribution of the vibrato to tone quality, leaving the term _timbre_ (simultaneous fusion) for the effect of overtone structure on tone quality.

The pitch of the pulsating tone is heard as "one salient pitch near the mean pitch of the oscillations in the vibrato tone." A trained observer can pick out "illusory upper and lower limits at will as continuous pitches," the salient pitch, however, remaining dominant. Musical observers ordinarily "estimate the extent of fluctuation to be about one fourth of what it actually is."

With these facts in mind, Seashore gave the following definition of the vibrato:

> The vibrato in music is a periodic pulsation, generally involving pitch, intensity, and timbre, which produces a pleasing flexibility, mellowness, and richness of tone.[23]

---

23. _University of Iowa Studies in the Psychology of Music_, vol. I, p. 349.

Not all musicians agree with this definition of the vibrato. Some regard it as a fault or defect to be avoided. Certainly the subject has been a confused one for many. The following quotation from Seashore offers some explanation:

> It is interesting to note that the real ground for the confusion which exists in the musical world and even among scientific and critical listeners lies in the fact that in the hearing of the vibrato, the normal ear is subject to a series of gross normal illusions. Among these is the astonishing under-estimation of the magnitude of the vibrato in hearing which lies in the fact that the oscillation of pitch is heard as if it were only ¼ to ½ of its actual extent, and a similar underestimate occurs for intensity. Another normal illusion consists in the persistent confusion of oscillations in pitch and intensity in listening so that a musician, or even a generation of musicians, will assert that it is oscillation in pitch, that it is oscillation in intensity, or, more frequently, that it is neither. A third normal illusion which makes the vibrato in its present gross form tolerable is the phenomenon of sonance, which lies in the fact that successive periodicities, when of sufficient rate, tend to fuse into a unified tone somewhat in the same manner that the simultaneous overtones in a violin clang fuse and are heard together as one tone. A fourth normal illusion which is a condition for making the vibrato tolerable is the fact that even with a pitch oscillation of a semitone the intonation is heard as of a particular tone which can easily be identified with standard pitch, the musical effect heard being that of a changing tone quality rather than specific changes in pitch, intensity, and timbre. Were it not for these four and numerous similar normal illusions which function in all musical hearing, the vibrato as it now exists would be utterly intolerable. It is this fact, that the vibrato is not heard even by the best musicians as it really is, which lies at the bottom of the confusion which has prevailed on this subject.[24]

A more recent study made at the University of Iowa investigated the preferences of listeners as to rate and

---

24. Ibid., p. 10.

extent of frequency fluctuations in the vibrato.[25] Complex tones were used in this experiment. It was found that some vibrato was definitely preferred to none; that the preferred rate was six to seven pulsations per second; that persons without musical training preferred an extent of a quarter tone; and that trained musicians preferred an extent of one tenth of a tone. We note that the average rate of pulsation and the preferred rate are the same and that the average extent of frequency variation is greater than the preferred extent.

---

25. John F. Corso and Don Lewis, "Preferred Rate and Extent of the Frequency Vibrato," Journal of Applied Psychology, vol. 34 (1950), pp. 206-212.

Chapter 2

THE AUDIBLE PHENOMENA OF HARMONIC INTERVALS

When two or more simple tones are presented to the ear simultaneously as in the acoustic spectrum of an interval, one might expect every tone to sound just as it does alone without any disturbance or other effect arising from the presence of the other tones, but such is not necessarily so. Any or all of the following may occur: (1) interference, (2) combination tones, (3) masking. In this chapter, separate consideration is given to each of these three phenomena.

A. Interference: Basic Formulation

Interference is either a reinforcement or a weakening of the sound of a harmonic interval that may result when any two tones of its spectrum are at or near the same frequency. The sound is reinforced when the tones are in the same phase but weakened when they are in opposite phase (a phase difference of π radians).

When two tones have the same frequency, their phase relationship is constant; and the interference, if any, is steady. Thus, two or more simple tones forming a coincidence by sounding together at the same frequency combine to produce one effective tone whose frequency is that of the component tones and whose intensity depends on the intensities and phase relations of the components.

When two tones have different frequencies, their phase relationship varies periodically between same and opposite. This results in alternate reinforcement and weakening; and the interference is unsteady, giving rise to an experience of pulsation. Consequently, two or more simple and therefore steady tones sounding together at slightly different frequencies have not only the aspect of a spectrum but also the aspect of a single pulsating tone, called the intertone, that is neither simple nor steady. The pulsations of the intertone are commonly known as beats and can be heard in the near approach to any interval that exhibits the coincidence of partial tones in its spectrum. In this section, the physical reality of these two apparently contradictory aspects of the same phenomenon is demonstrated mathematically. The word intertone is used to denote either the effective tone when the spectral tones are at the same frequency or the pulsating tone when the spectral tones are at different frequencies.

The following symbols pertain to the tones of a spectrum:

n = the number of tones in the spectrum. $\geqq 2$.

$r, s = 1, 2, 3, \ldots n$
= integers denoting tones of the spectrum.

x_r = frequency in cycles per second. Constant.
$0 \leqq x_1 \leqq x_2 \leqq x_3 \leqq \cdots \leqq x_n$

$\Omega_r = 2\pi x_r$
= frequency in radians per second. Constant.

l_r = zero or a positive or negative integer or integral variable.

$P_r = \Omega_r(t - T_r) - 2\pi l_r = 2\pi[x_r(t - T_r) - l_r]$
$= \Omega_r t + q_r$
= the phase angle at time t. Variable.

$q_r = -(\Omega_r T_r + 2\pi l_r) = -2\pi(x_r T_r + l_r)$
= phase displacement. Constant.
= the phase angle at time 0. Constant.

T_r = the first time when $P_r = 0$ or a multiple of 2π. Constant.

V_r = velocity amplitude. Constant.

$I_r = \frac{1}{2} D c V_r^2$ = intensity. Constant.

The symbols pertaining to the intertone are as follows:

P = phase angle. Variable.

$\Omega = dP/dt$
= the instantaneous frequency in radians per second. Constant or variable.

$x = \Omega/2\pi$
= the instantaneous frequency in cycles per second. Constant or variable.

V = velocity amplitude. Variable.

$I = \frac{1}{2} DcV^2$
= the instantaneous intensity. Variable.

The particle velocity of the air in the external auditory canal is given as resulting from a spectrum by

$$v = \sum_{r=1}^{n} V_r \cos P_r \qquad \text{A1}$$

The same particle velocity is given as resulting from the intertone by

$$v = V \cos P \qquad \text{A2}$$

These relations for n = 4 are illustrated in Figure 1, whence we obtain two more equations:

$$w = \sum_{r=1}^{n} V_r \sin P_r \qquad \text{A3}$$

$$w = V \sin P \qquad \text{A4}$$

These four equations are now employed to determine the intensity, phase angle, and frequency of the intertone in terms of the spectrum.

$$v^2 = V^2 \cos^2 P = \sum_{r=1}^{n} \sum_{s=1}^{n} V_r V_s \cos P_r \cos P_s \qquad \text{A5}$$

$$w^2 = V^2 \sin^2 P = \sum_{r=1}^{n} \sum_{s=1}^{n} V_r V_s \sin P_r \sin P_s \qquad \text{A6}$$

$$v^2 + w^2 = V^2 = \sum_{r=1}^{n} \sum_{s=1}^{n} V_r V_s \cos (P_s - P_r) \qquad \text{A7}$$

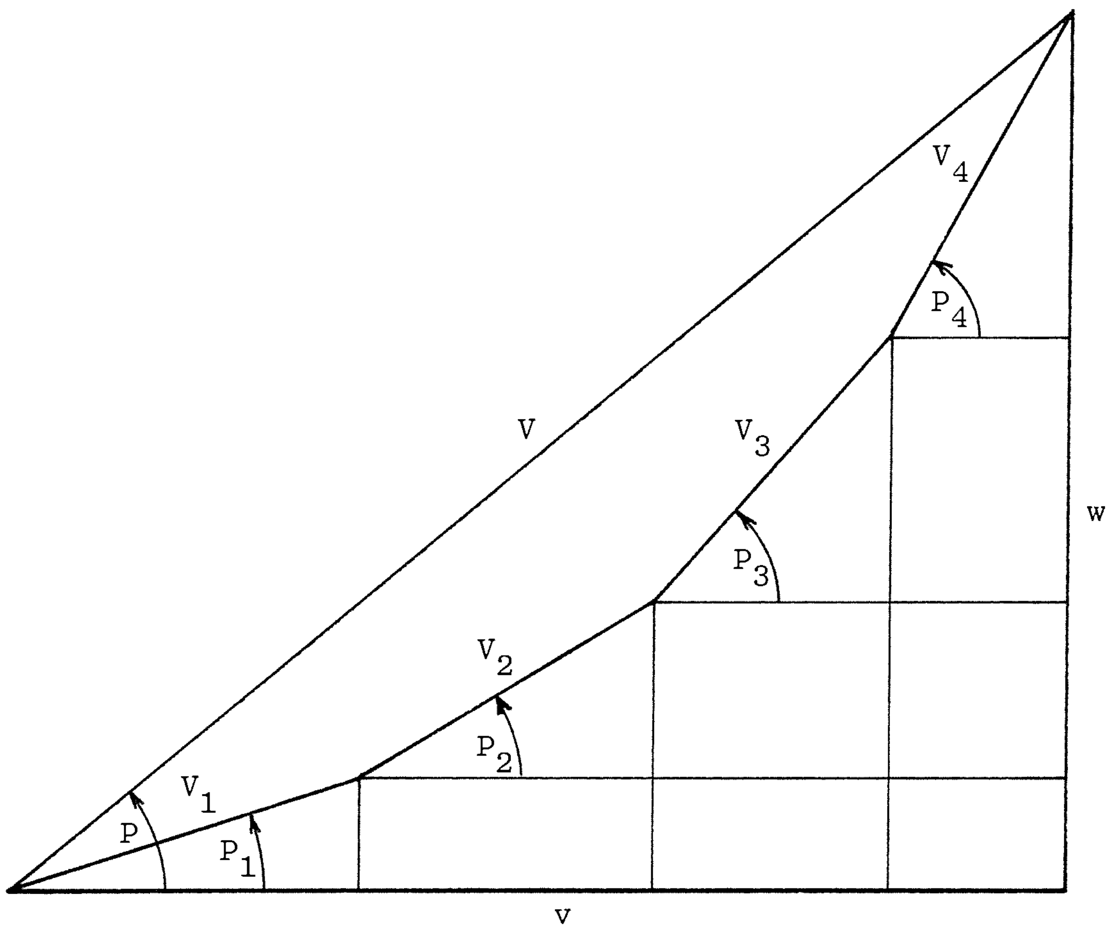


Figure 1. Relations between the spectrum and the intertone.

$$\begin{aligned} V^2 &= V_1[V_1 + V_2 \cos(P_2 - P_1) + V_3 \cos(P_3 - P_1) + \ldots] \\ &\quad + V_2[V_1 \cos(P_2 - P_1) + V_2 + V_3 \cos(P_3 - P_2) + \ldots] \\ &\quad + V_3[V_1 \cos(P_3 - P_1) + V_2 \cos(P_3 - P_2) + V_3 + \ldots] \\ &\quad + \ldots \end{aligned} \quad \text{A8}$$

$$\begin{aligned} &= V_1^2 + V_2^2 + V_3^2 + \ldots + 2V_1V_2 \cos(P_2 - P_1) \\ &\quad + 2V_1V_3 \cos(P_3 - P_1) + 2V_2V_3 \cos(P_3 - P_2) + \ldots \end{aligned} \quad \text{A9}$$

$$I = \tfrac{1}{2} DcV^2$$

$$\begin{aligned} &= I_1 + I_2 + I_3 + \ldots + 2\sqrt{I_1I_2} \cos(P_2 - P_1) \\ &\quad + 2\sqrt{I_1I_3} \cos(P_3 - P_1) + 2\sqrt{I_2I_3} \cos(P_3 - P_2) + \ldots \end{aligned} \quad \text{A10}$$

= the instantaneous intensity of the intertone.

$$w/v = \sin P/\cos P = \tan P \qquad \text{A11}$$

$$P = \arc\tan (w/v) \qquad \text{A12}$$

= the phase angle of the intertone.

$$dv/dt = -\sum_{r=1}^{n} \Omega_r V_r \sin P_r \qquad \text{A13}$$

$$dw/dt = \sum_{r=1}^{n} \Omega_r V_r \cos P_r \qquad \text{A14}$$

$$vdw/dt = \sum_{r=1}^{n} \sum_{s=1}^{n} \Omega_r V_r V_s \cos P_r \cos P_s \qquad \text{A15}$$

$$wdv/dt = -\sum_{r=1}^{n} \sum_{s=1}^{n} \Omega_r V_r V_s \sin P_r \sin P_s \qquad \text{A16}$$

$$\Omega = dP/dt = \frac{vdw/dt - wdv/dt}{v^2 + w^2} \qquad \text{A17}$$

$$= \sum_{r=1}^{n} \sum_{s=1}^{n} \Omega_r V_r V_s \cos (P_s - P_r)/V^2 \qquad \text{A18}$$

= the instantaneous frequency of the intertone in radians per second.

$$x = \Omega/2\pi$$

$$= \sum_{r=1}^{n} \sum_{s=1}^{n} x_r V_r V_s \cos (P_s - P_r)/V^2 \qquad \text{A19}$$

= the instantaneous frequency of the intertone in cycles per second.

$$x = \frac{\begin{array}{l} x_1V_1[V_1 + V_2 \cos (P_2 - P_1) + V_3 \cos (P_3 - P_1) + \ldots] \\ + x_2V_2[V_1 \cos (P_2 - P_1) + V_2 + V_3 \cos (P_3 - P_2) + \ldots] \\ + x_3V_3[V_1 \cos (P_3 - P_1) + V_2 \cos (P_3 - P_2) + V_3 + \ldots] \\ + \ldots \end{array}}{V^2} \qquad \text{A20}$$

$$x = \frac{\begin{array}{l} x_1V_1^2 + x_2V_2^2 + x_3V_3^2 + \ldots + (x_1 + x_2)V_1V_2 \cos (P_2 - P_1) \\ + (x_1+x_3)V_1V_3 \cos (P_3-P_1) + (x_2+x_3)V_2V_3 \cos (P_3-P_2)+\ldots \end{array}}{V^2}$$

A21

$$x = \frac{x_1I_1 + x_2I_2 + x_3I_3 + \ldots + (x_1+x_2)\sqrt{I_1I_2}\cos(P_2-P_1) + (x_1+x_3)\sqrt{I_1I_3}\cos(P_3-P_1) + (x_2+x_3)\sqrt{I_2I_3}\cos(P_3-P_2) + \ldots}{I} \qquad \text{A22}$$

The terms in A10 that contain the functions $\cos(P_2 - P_1)$, $\cos(P_3 - P_1)$, $\cos(P_3 - P_2)$, and so on [that is, terms of the type $2\sqrt{I_rI_s}\cos(P_s - P_r)$, $r < s$] arise from pairs of tones in the spectrum and account for the interference that may result when the tones of a pair are at or near the same frequency. Since terms of this type account for the interference, they are referred to here as interference terms.

$$P_s - P_r = (\Omega_s - \Omega_r)t + q_s - q_r \qquad \text{A23}$$

= the phase difference between any two tones of the spectrum.

When the tones of a pair have the same frequency, $\Omega_r = \Omega_s$, $P_s - P_r = q_s - q_r$ = a constant, and the corresponding interference term is a constant. If all the tones of the spectrum have the same frequency, all the interference terms are constants, I is constant, and the right side of A22 reduces to the common frequency of the spectral tones, showing that the intertone is then effectively just one simple tone. Were it not for the interference, the intensity of the effective tone would equal the sum of the intensities of the components.

When the tones of a pair have different frequencies, $P_s - P_r$ is not a constant, and the corresponding interference term varies sinusoidally, thereby giving rise to a beating of the intertone. Each such term has a maximum value of $2\sqrt{I_rI_s}$, a minimum of $-2\sqrt{I_rI_s}$, and an average of 0. These terms subtract as much as they add to the fluctuating value of I; therefore, the average value of I is obtained by simply deleting these terms. Let this be designated by H;

then

$$H = \sum_{r=1}^{n} I_r \qquad \text{A24}$$

= the average intensity of the intertone.

The beats consist in the periodic variation of I from its average; and $2\sqrt{I_r I_s}$ are the amplitudes of this variation; therefore, the $2\sqrt{I_r I_s}$ are measures of the physical intensity of the beats.

Averaging the product of A22 and I/H results in a weighted average value of x that counts the more intense cycles of the intertone as if they were more numerous instead of more intense. This may be called the salient frequency of the intertone and is designated by

$$S = \sum_{r=1}^{n} x_r I_r / H \qquad \text{A25}$$

That x_1 and x_n are lower and upper limits of S can be shown as follows:

$$S = x_s + \sum_{r=1}^{n} (x_r - x_s) I_r / H \qquad \text{A26}$$

$$= x_1 + \sum_{r=1}^{n} (x_r - x_1) I_r / H > x_1 \qquad \text{A27}$$

$$= x_n - \sum_{r=1}^{n} (x_n - x_r) I_r / H < x_n \qquad \text{A28}$$

$$x_1 < S < x_n \qquad \text{A29}$$

Inasmuch as a sum is independent of the order of summation, A28 may be written as

$$S = x_n - \sum_{r=1}^{n} (x_n - x_{n+1-r}) I_{n+1-r} / H \qquad \text{A30}$$

If the spectrum is balanced,

$$x_n - x_{n+1-r} = x_r - x_1 \qquad \text{A31}$$

$$I_{n+1-r} = I_r \qquad \text{A32}$$

A30 becomes

$$S = x_n - \sum_{r=1}^{n} (x_r - x_1) I_r / H \qquad \text{A33}$$

and addition of this to A27 results in

$$2S = x_1 + x_n$$

whence

$$S = \tfrac{1}{2} (x_1 + x_n) \qquad \text{A34}$$

$$= \tfrac{1}{2} (x_r + x_{n+1-r}) \qquad \text{A35}$$

The author suggests that, even as the pitch of a pulsating tone is heard as "one salient pitch near the mean pitch of the oscillations in the vibrato tone,"[1] an experimental investigation would find one salient pitch attributed to an intertone that would correlate well with the salient frequency S.

B. Interference: Beats

The rate of increase of $P_s - P_r$ in radians per second is

$$d(P_s - P_r)/dt = \Omega_s - \Omega_r \qquad \text{B1}$$

and the frequency of cos $(P_s - P_r)$ in cycles per second is

$$[d(P_s - P_r)/dt]/2\pi = x_s - x_r \qquad \text{B2}$$

A10 and A22 contain terms that are proportional to cos $(P_s - P_r)$; therefore, I and x pulsate at such frequencies (or rates) as $x_s - x_r$, $r < s$. In other words, the frequencies of the interference terms are the rates of pulsation of the intertone, and these rates of pulsation are the differences between the frequencies of the spectrum.

If no two frequency differences are the same, we find the following correspondence between the number of tones in the spectrum and the number of rates of pulsation in the intertone:

---

1. Ch. 1, Sec. H.

Number of tones	2	3	4	5	6	n
Number of rates	1	3	6	10	15	n(n - 1)/2

However, if some frequency differences are the same, the number of rates of pulsation is less. In particular, if there is just one common frequency difference between successive tones of the spectrum, the correspondence is as follows:

Number of tones	2	3	4	5	6	n
Number of rates	1	2	3	4	5	n - 1

In this case, the common difference between successive frequencies is the lowest (or fundamental) rate of pulsation, and all other frequency differences are multiples of this as in the harmonic series. The result is that the pulsations, taken together, are periodic. Let Δt equal the period of one pulsation, that is, the increase of t in one pulsation. Then the fundamental rate of pulsation (the number of pulsations per second) is $1/\Delta t$; and other rates, if any, are $2/\Delta t$, $3/\Delta t$, and so forth. The restriction to one common difference between successive frequencies and the immediate consequences of this restriction can be expressed in a general way by the simple statement that

$$x_s - x_r = (s - r)/\Delta t \qquad \text{B3}$$

A7 and A12 are necessary and sufficient conditions for A2 to be identically equal to A1, but they are not sufficient for a spectrum of steady tones to be physically likened to and heard as a pulsating tone. For example, a steady complex musical tone is an acoustic spectrum of simple tones (evident by comparison of A1 with G10 in Chapter 1) and can be mathematically represented by A2; but it is not realistic to conceive of it as a pulsating tone, and it is certainly not heard as such. Its intensity is the same as the average intensity it would have if regarded as a pulsating intertone, but the pitch attributed to it corresponds to the lowest tone of the spectrum rather than the salient frequency of the intertone.

Every term in the numerator of A22 is matched by a term in its denominator (A10) that involves the same tone or tones of the spectrum; and, since division of the numerator by the denominator gives the frequency of the intertone, division of each term of the numerator by the matching term of the denominator gives component frequencies of the intertone such that the frequency of the intertone is at every instant an interpolation or extrapolation of the component frequencies. The component frequencies thus obtained are x_1, x_2, x_3, ... and $(x_1 + x_2)/2$, $(x_1 + x_3)/2$, $(x_2 + x_3)/2$, ... These fall into two groups: the first being the frequencies x_r of the spectral tones, and the second being averages $(x_r + x_s)/2$ of the frequencies of pairs of spectral tones. Those in the first group enter into the calculation of the salient frequency (A25); those in the second group are associated with the interference terms, each average frequency being connected with the interference term that involves the same two spectral tones.

The component frequency associated with the interference term involving tones r and s is $\frac{1}{2}(x_r + x_s)$; and the rate of beating of the same interference term is $x_s - x_r$; therefore, the number of cycles of the component frequency for each beat of this term is the quotient $\frac{1}{2}(x_r + x_s)/(x_s - x_r)$. If this quantity is of sufficient magnitude, every beat is well defined as a tone, and the intertone is well defined as a pulsating tone. If $\frac{1}{2}(x_r + x_s)/(x_s - x_r)$ is too small, the needed repetition of cycles within a beat is lacking and beats are not heard.

This requirement of a minimum number of cycles per beat to give the sound of a series of separate pulses of tone is essentially the same as the requirement noted in Ch. 1, Sec. E, of a minimum number of cycles to convey the sensation of tone. The few cycles required just to evoke sensations of pitch and loudness are not enough to establish the pitch and loudness finally experienced after exposure to many cycles. There is however a limit to the number of cycles needed to determine the attributes of a tone with a feeling of finality. Thus there are two limits to the number of cycles related to establishing the attributes of a tone: (1) a lower limit required to evoke sensations of tone and (2) an upper limit beyond which the recognition of these attributes is not improved. For example, certain experimental results have indicated that 10 cycles of a 1000-cps tone convey a sensation of pitch but that about 100 cycles are required to determine its pitch with reasonable certainty. The lower limit can be set with more precision than the upper limit, and these limits may vary with the frequency. Between the two limits, the pitch and loudness build up to their ultimate values gradually as the number of cycles increases.

Similar considerations relate the perception of beats to the number of cycles per beat. If there are more than two tones in the spectrum, there are two or more values of $\frac{1}{2}(x_r + x_s)/(x_s - x_r)$; and, within limits, the beats with more cycles per beat are better defined than those with less. The reciprocal of $\frac{1}{2}(x_r + x_s)/(x_s - x_r)$ is

$$2(x_s - x_r)/(x_r + x_s) = 2(x_s/x_r - 1)/(x_s/x_r + 1) \cong \ln(x_s/x_r) \qquad \text{B4}$$

This is a good approximation as long as $2(x_s - x_r)/(x_r + x_s)$ is not too large (or when $\frac{1}{2}(x_r + x_s)/(x_s - x_r)$ is not too small), which is the condition under which beats can be heard. The usefulness of this relationship becomes apparent when we recall that the size of an interval is directly proportional to the logarithm of its frequency ratio (Ch. 1, Sec. B). Thus, within the limits of interest to us, the number of cycles per beat is approximately equal to $1/\ln(x_s/x_r)$, which equals 17.31 divided by the size of the interval between tones r and s in semitones. Clearly then, since the number of cycles per beat is inversely proportional to the size of the interval, within limits, the beats between the smaller intervals are better defined than those between the larger intervals, and no beats are heard between intervals that are too large.

This relationship between the number of cycles per beat and the size of the interval between the beating tones makes it easy for musicians to determine for themselves the approximate number of cycles per beat required to give audible beats even when working with complex tones provided intervals greater than a second are in just intonation. It is found that, of the intervals in common use, the minor second gives the most definite beats, the major second also gives definite beats altho less so, and the minor third gives beats only in the bass range. Upon investigating the effect of intervals between the major second and the minor third in size and in the middle range, the author experiences beats when the frequency ratio between two beating tones is 8/7 but is uncertain when the ratio is 7/6. This finding places the requirement for audible beats at approximately 7.5 or more cycles per beat (or an interval of 2.31 semitones or less).

It is easily understood now why a steady complex musical tone is not heard as a pulsating tone. Its fundamental frequency is x, and the frequencies of the partials are multiples of x. Let $x_r = rx$ and $x_s = sx$; then $x_s - x_r = (s-r)x$, $x_s/x_r = s/r$, and $\frac{1}{2}(x_r + x_s)/(x_s - x_r) = \frac{1}{2}(r + s)/(s - r)$. Beats, if any, must occur between the tones of the smaller intervals, and these occur when $s = r + 1$, in which case the number of cycles per beat is $r + \frac{1}{2}$. It is immediately seen that the intervals between the lower and usually stronger partials (in which $r < 7$) are much too large to give rise to beats. Beats between the higher successive and usually weaker partials (in which $r \geqq 7$) have the same rate of beating as the fundamentlal frequency of the tone itself and are likely to be accepted as merely a contribution to the timbre. This matter was investigated by Helmholtz, who said:

> If the 15th and 16th partials of a compound tone are still audible, they form the interval of a semitone, and naturally produce the cutting beats of this dissonance. That it is really the beats of these tones which cause the roughness of the whole compound tone can be easily felt by using a proper resonator. ...
>
> Hence there can no longer be any doubt that motions of the air corresponding to deep musical tones compounded of numerous partials are capable of exciting at one and the same time a continuous sensation of deep tones and a discontinuous sensation of high tones, and become rough or jarring through the latter. Herein lies the explanation of the fact already observed in examining qualities of tone that compound tones with many high upper partials are cutting, jarring, or braying; and also of the fact that they are more penetrating and cannot readily pass unobserved, for an intermittent impression excites our nervous apparatus much more powerfully than a continuous one, and continually forces itself afresh on our perception. On the other hand, simple tones or compound tones which have only a few of the lower upper partials, lying at wide intervals apart, must produce perfectly continuous sensations in the ear, and make a soft and gentle impression, without much energy, even when they are in reality relatively strong.[2]

As Helmholtz noted, beats produce a sensation of roughness (jarring or shaking) in the ear. The degree of roughness depends on (1) the number of cycles per beat, (2) the component (or center) frequency $\frac{1}{2}(x_r + x_s)$, and (3) the

---

2. Sensations of Tone, pp. 178-179. The expression "compound tone" means a complex tone.

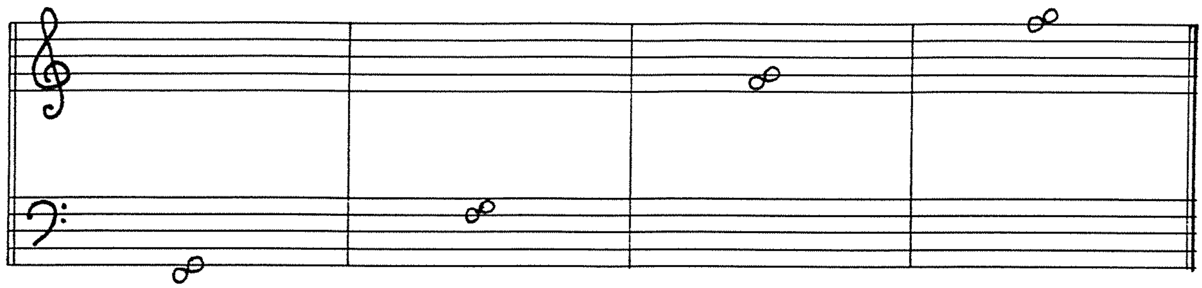


Figure 2. Decreasing roughness with increasing center frequency and a constant 8.7 cycles per beat.

intensity $2\sqrt{I_r I_s}$ of the beats. Given constant intensity and a constant center frequency in the middle range, the roughness increases as the number of cycles per beat increases from about 7.5 to 50. Like the number of cycles needed to convey the sensation of tone,[3] the number of cycles per beat needed to give definite beats increases with the frequency; therefore, the roughness of a given small interval ordinarily decreases as its center frequency increases (Figure 2) because the number of cycles per beat, being determined by its size, does not increase as would be needed to keep the same degree of roughness.

In an interval composed of complex tones, let x be the frequency of the higher tone and y be that of the lower. Also, let $x°$ and $y°$ be a pair of small relatively prime integers with $x° \geqq y°$. Then, if $x/y = x°/y°$, the relations exist that are described in connection with D1 and D2 of Chapter 1. If, to a slight extent, $x/y > x°/y°$, then $y°x > x°y$, and the partial $y°x$ of the higher tone is thus seen to be a little higher than the partial $x°y$ of the lower tone. Furthermore, the frequency ratio of these two partials is

$$(y°x)/(x°y) = (x/y)/(x°/y°) \qquad \text{B5}$$

which shows that the interval between the two partials is equal to the difference between the intervals x/y and $x°/y°$.

3. Stevens and Davis, <u>Hearing</u>, pp. 101-102.

If this difference is small enough ($\leqq$ 2.31 semitones), and if the partials are of sufficient intensity, then beats are heard at the rate of $y°x - x°y$ per second when tones x and y are sounded together.

Inasmuch as $(2y°x)/(2x°y) = (y°x)/(x°y)$, the same interval exists between the partials $2y°x$ and $2x°y$, and beats are heard between them too if their intensity permits; but, since $2y°x - 2x°y = 2(y°x - x°y)$, these beats are twice as fast as those between $y°x$ and $x°y$. Likewise, beats may be heard between $3y°x$ and $3x°y$ at the rate of $3(y°x - x°y)$ per second. We see from this that $y°x - x°y$ may appropriately be called the fundamental rate of beating of the interval x/y and that the other rates of beating are integral multiples of this. As may be observed in Table I, Ch. 1, the higher partials are ordinarily weaker than the lower ones; therefore, since the faster beats arise from the higher partials, only the fundamental rate of beating is usually noticed.

Recalling the rule given in Ch. 1, Sec. D, concerning the coincidence of partials, and observing that the partial tones that coincide when x/y equals $x°/y°$ differ slightly in frequency when x/y differs slightly from $x°/y°$, we see that these beats result from the approximation of just intervals. If slow enough, these beats are not necessarily disagreeable; but they do betray proximity of the interval sounded to a just interval and have doubtless influenced musicians, thru a desire to avoid them, to prefer just intonation or, at least, to regard it as the standard of correct intonation. Tuning an interval by eliminating beats is a common practice, and it leads to a high degree of accuracy in performance. Extensive use of beats is made in tuning pianos and organs altho most of the intervals of the equally tempered scale, being slightly "out of tune," must be tuned so as to give rise to some degree of beating rather than to eliminate beating altogether.

C. Modulation

The pulsations of the intertone resemble a vibrato in that they consist of synchronous pulsations of both frequency and intensity (Ch. 1, Sec. H). They can be made to resemble a vibrato in yet other respects by the imposition of appropriate conditions on the spectrum. Conformity to B3 makes the pulsations periodic, and setting Δt equal to .15 seconds places the rate of pulsation between 6 and 7 per second. In addition to this, proper choice of the number, intensities, and phase relations of the spectral tones establishes the extent and form of variation of both frequency and intensity.

The similarity between the pulsations of the intertone and a vibrato suggests that any tone performed with a vibrato is physically equivalent to a spectrum of steady tones. This aspect of the vibrato was pointed out in 1931 by Shower and Biddulph[4] and given a fuller discussion in 1938 by Stevens and Davis.[5] It was easily demonstrated that a tone with a constant frequency and a sinusoidally modulated amplitude (that is, with an intensity vibrato) is physically equivalent to three steady tones with a fundamental frequency difference equal to the rate of modulation. On the other hand, it was found that a tone with a constant amplitude and sinusoidally modulated frequency could only be approximated by a finite number of steady tones. The following

4. "Differential Pitch Sensitivity of the Ear," The Journal of the Acoustical Society of America, vol. 3 (1931), pp. 275-287.

5. Hearing, Chapter 9 and Appendix I.

mathematical demonstration is pertinent.

$$\sum_{r=1}^{n} V_r \cos (P - P_r)$$

$$= \sum_{r=1}^{n} V_r (\cos P_r \cos P + \sin P_r \sin P)$$

$$= v \cos P + w \sin P \qquad \text{by A1 and A3}$$

$$= V(\cos^2 P + \sin^2 P) \qquad \text{by A2 and A4}$$

$$= V \qquad \text{C1}$$

Likewise,

$$\sum_{r=1}^{n} V_r \sin (P - P_r) = 0 \qquad \text{C2}$$

C1 can also be written

$$V = \sum_{r=1}^{n} V_r \cos (P - P_r) \qquad \text{C3}$$

and differentiation results in

$$dV/dt = \sum_{r=1}^{n} (\Omega_r - \Omega) V_r \sin (P - P_r)$$

which, because of C2, simplifies to

$$dV/dt = \sum_{r=1}^{n} \Omega_r V_r \sin (P - P_r) \qquad \text{C4}$$

To have frequency modulation without amplitude modulation would require that V be constant, whence dV/dt would equal 0, but C2 and C4 cannot both be 0 at all times; therefore, a spectrum of steady tones cannot give rise to frequency modulation without amplitude modulation.

On the other hand, it is easily possible to have amplitude modulation without frequency modulation. For example, let n = 5 and let $P = P_3$, which has the constant derivative Ω_3. Then, by C2,

$$\sum_{r=1}^{5} V_r \sin(P_3 - P_r) = 0$$

This is true at all times if the spectrum is balanced about its central tone (r = 3), that is, if

$$V_2 = V_4$$

$$V_1 = V_5$$

$$P_3 - P_2 = P_4 - P_3 \qquad \text{C5}$$

$$P_3 - P_1 = P_5 - P_3 \qquad \text{C6}$$

Then, by C3,

$$V = \sum_{r=1}^{5} V_r \cos (P_3 - P_r)$$

$$= V_3 + 2V_4 \cos (P_4 - P_3) + 2V_5 \cos (P_5 - P_3) \qquad \text{C7}$$

In view of A23, C5 becomes

$$(\Omega_3 - \Omega_2)t + q_3 - q_2 = (\Omega_4 - \Omega_3)t + q_4 - q_3 \qquad \text{C8}$$

Since this must be true at all times,

$$\Omega_3 - \Omega_2 = \Omega_4 - \Omega_3 \qquad \text{C9}$$

$$x_3 - x_2 = x_4 - x_3 \qquad \text{C10}$$

$$q_3 - q_2 = q_4 - q_3 \qquad \text{C11}$$

Similar relationships regarding the first, third, and fifth tones of the spectrum follow from C6.

The possibility of a tone with an intensity vibrato having a spectrum of steady tones can be demonstrated in another way. Let n = 3 and subtract x_2 from both sides of A22 making use of the fact that the denominator of the right side of A22 is as given in A10. The result is

$$x - x_2 = [(x_1 - x_2)I_1 + (x_3 - x_2)I_3 + (x_1 - x_2)\sqrt{I_1 I_2} \cos (P_2 - P_1)$$
$$+ (x_1 + x_3 - 2x_2)\sqrt{I_1 I_3} \cos (P_3 - P_1)$$
$$+ (x_3 - x_2)\sqrt{I_2 I_3} \cos (P_3 - P_2)]/I \qquad \text{C12}$$

This can be simplified by the introduction of B3, which means that

$$x_3 - x_2 = x_2 - x_1 = 1/\Delta t \qquad \text{C13}$$

$$x_1 + x_3 = 2x_2 \qquad \text{C14}$$

and, in place of C12, gives us

$$x = x_2 + \frac{I_3 - I_1 + \sqrt{I_2 I_3}\cos(P_3 - P_2) - \sqrt{I_1 I_2}\cos(P_2 - P_1)}{I\Delta t} \qquad \text{C15}$$

To complete this demonstration, we set $I_3 = I_1$ and $P_3 - P_2 = P_2 - P_1$. Then $x = x_2$, which is constant, and the vibrato affects only the intensity.

Comparable results are obtained when the spectrum consists of only two simple tones. From A10 and A22 with $n = 2$ and $x_2 - x_1 = 1/\Delta t$,

$$x = x_1 + [I_2 + \sqrt{I_1 I_2}\cos(P_2 - P_1)]/(I\Delta t) \qquad \text{C16}$$

$$= x_2 - [I_1 + \sqrt{I_1 I_2}\cos(P_2 - P_1)]/(I\Delta t) \qquad \text{C17}$$

$$= (x_1 + x_2)/2 + (I_2 - I_1)/(2I\Delta t) \qquad \text{C18}$$

Setting $I_2 = I_1$ is sufficient to make $x = (x_1 + x_2)/2$, a constant, restriction of phase being unnecessary. This condition also makes

$$2\sqrt{I_1 I_2} = 2I_1 = I_1 + I_2 = H \qquad \text{C19}$$

$$I = H\,[1 + \cos(P_2 - P_1)] \qquad \text{C20}$$

which means that, in this case, I reaches a maximum of 2H and a minimum of 0. The intensity of the intertone cannot reach a maximum greater than twice its average or a minimum less than zero; therefore, for a certain average intensity of the intertone, the strongest beats are obtained when $I_2 = I_1$. This is the condition commonly referred to as "best beats."

D. Aural Harmonics and Combination Tones

Steady tones of sufficient intensity generate an aural response of four classes of tones distinguished by their frequency relations. The frequencies of the first class (primary tones) are identical to those of the generating tones (or generators), the frequencies of the second class (aural harmonics) are multiples of those of the first class, the frequencies of the third class (difference tones) are differences between those of the first and second classes, and the frequencies of the fourth class (summation tones) are sums of those of the first and second classes. A combination tone is a tone of the third or fourth class. The entire assemblage of tones contained in the response of the ear constitutes what we call the aural spectrum, and the tones thereof are called spectral tones or response tones. The aural harmonics and combination tones were formerly referred to as subjective tones; but the explanation as to how the ear produces them rests on purely physical grounds; therefore, the term "subjective" is no longer favored for such tones. They will be referred to here as secondary tones to distinguish them from the primary tones.

Let us now consider the aural responses to one or two pure tones. Responses to complex tones will be considered later. One pure tone of frequency x generates a primary tone of frequency x and aural harmonics of frequencies 2x, 3x, and so forth. A pure tone by itself does not generate combination tones. Two pure tones of frequencies x and y, with $x > y$, generate two primary tones of frequencies x and y; aural harmonics of frequencies 2x, 2y, 3x, 3y, ...; difference tones of frequencies $x - y$, $2y - x$, $2x - y$, $2x - 2y$, ...; and summation tones of frequencies $x + y$, $2y + x$, $2x + y$, $2x + 2y$, ... These constitute all the classes of tones in the aural spectrum.

Not only are the spectral tones identified by the algebraic expressions for their frequencies, but also degrees of relationship of these tones to the generators can be seen in these expressions. The primary tones can be said to be in the first degree of relationship, or they can be called tones

Table I

Aural Spectrum Generated by Two Pure Tones

The algebraic expression for the frequency of a spectral tone not only indicates its frequency, class, and degree but also uniquely identifies the relationship of the individual tone to the generators.

Degree	1. Primary tones	Secondary tones		
		2. Aural harmonics	Combination tones	
			3. Difference tones	4. Summation tones
1	x y			
2		2x 2y	x - y	x + y
3		3x 3y	2y - x 2x - y	2y + x 2x + y
4		4x 4y	2x - 2y 3y - x 3x - y	2x + 2y 3y + x 3x + y

of the first degree because their frequencies are identical to those of the generators. The degree of an aural harmonic is given by the coefficient of x or y, and the degree of a combination tone is given by the sum of the coefficients of x and y. For example, 2x - y identifies a tone of the third degree because the coefficient of x is 2, that of y is 1, and their sum is 3. Table I shows the spectral tones thru the fourth degree that are generated by two pure tones of frequencies x and y.

Some of the tones of the aural spectra of the major sixth and the major third in just intonation (frequency

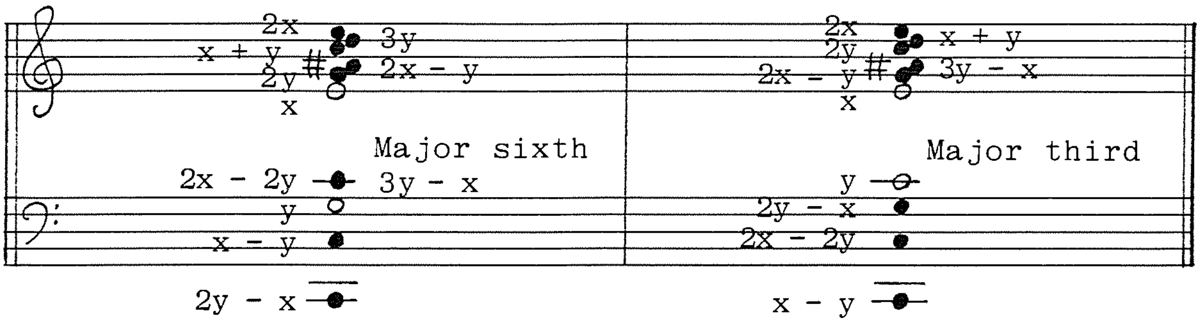


Figure 3. Some of the tones of the aural spectra generated by pure tones of frequencies x and y in the intervals of the major sixth and the major third. The white notes represent the primary tones; the black notes, the secondary tones.

ratios 5/3 and 5/4) are shown in Figure 3. Both spectra have the frequency ratios of the harmonic series, but they arise from different stimuli and are known to sound different. We conclude, then, that they must have different loudness profiles. Differences in phase relations must also exist, but we can expect them to be relatively unimportant.

The loudness profile of an aural spectrum consists of the relative loudnesses of the tones of the spectrum. It is essentially a description of an auditory sensation, but only limited knowledge of a loudness profile can be obtained by direct observation. The hearing can easily recognize that one spectrum is different from another without being able to give a detailed description of either.

In order to supplement the testimony of the unaided ear, recourse has been had to an exploring (or auxiliary) tone, that is, a pure tone whose frequency, intensity, and phase can be varied and controlled so as to seek out and match or approximate any tone of a spectrum. When the frequency of the exploring tone matches or approximates that of a spectral tone, the observer can use the effects of interference so as to match or approximate the intensity of the spectral tone. The intensity of the exploring tone is then indicative of or equal to that of the spectral tone. Applying this procedure to all the tones of a spectrum gives us what we can call the intensity profile of an aural spectrum.

Table II presents an intensity profile obtained by Chesney R. Moe,[6] who had an observer adjust the intensity of an exploring tone until it produced the "best beats"[7] against a spectral tone and then assumed the intensity of the exploring tone to equal that of the spectral tone. The generators were practically pure tones of 950 and 690 cps, creating an interval of 5.54 semitones. Comparison of Table II with Table I shows that the aural harmonics and difference tones of lower degree have the greater intensity. The summation tones are weaker and less regular.

The experimental findings of many investigators[8] regarding loudness and intensity relations in an aural spectrum generated by two pure tones are summarized below:

1. The primary tones are louder than the secondary tones.
2. Secondary tones of a given class and a given degree (not necessarily the same as the class) have approximately the same intensity.
3. Among secondary tones of a particular class, those of the second degree are stronger than those of the fourth degree, those of the third degree are stronger than those of the fifth degree, and those of any degree are stronger than those of higher degrees when the generating tones are not too intense.

---

6. "An Experimental Study of Subjective Tones Produced Within the Human Ear," The Journal of the Acoustical Society of America, vol. 14 (1942), pp. 159-166.

7. See the end of Section C.

8. Helmholtz, Fletcher, Stevens and Davis, Moe, Plomp, and others cited by these.

Table II

Intensity Profile of an Aural Spectrum
Generated by two Pure Tones

The height of a tone in the column corresponds to its intensity level. Two tones of the same intensity are separated by a comma. (Adapted from Moe)

Intensity level in db	1. Primary tones	2. Aural harmonics	3. Difference tones	4. Summation tones
90				
	x, y			
85				
80				
75			x - y	
70		2x	2x - y	
			2y - x	
65		2y, 3x		
		3y	3y - x	2x + y
60			3x - y	2y + x
			2x - 2y	
		4x		
55		4y		
				4y + x
				x + y, 2x + 2y
50			4y - x	3x + y
			4x - y	3y + x
		5x, 5y		
45			3x - 2y	
				4x + y
				3x + 2y
40				3y + 2x

4. Among secondary tones of a particular degree, difference tones are stronger than aural harmonics and summation tones.
5. Increasing the intensity of the generating tones increases the intensity of the spectral tones; and, when such an increase occurs, the increase of the secondary tones is proportionally greater than the increase of the primary tones.
6. Pure tones must have intensity levels of about 50 db or greater to generate detectable secondary tones.
7. Difference tones are less easily heard when they lie between the primaries.

E. Aural Responses to Complex Tones

The partials of a complex tone present the ear with many generating tones; but, like one pure tone of sufficient intensity, a complex tone generates an aural spectrum all of whose frequencies are multiples of the fundamental, for the simple reason that the generators themselves are multiples of the fundamental. The partials of a complex tone of frequency x generate the aural spectrum shown in Table III. In the expression for the frequency of each spectral tone, the frequency of a partial is placed in parentheses when it is multiplied by a number other than 1, in order to preserve its identity as a generator and to make the degree of the tone visibly evident. As expected, all the expressions for the frequencies of tones of the second and third degrees simplify to first-degree expressions, that is, to multiples of x.

Not all of the partials enter into the generation of every response tone. A primary tone or an aural harmonic has one generating tone. The number of tones that act together to generate one combination tone is at least two but not greater than the degree of the tone. Thus, in Table III, two partials generate a combination tone of the second

Table III

Aural Spectrum Generated by a Complex Tone

The frequency of a partial is placed in parentheses when it is multiplied by a number other than 1. All the tones in a line have the same frequency.

Degree		
1	2	3
x	2x - x, 3x - 2x, 4x - 3x	3x - 2(x), 2(2x) - 3x
2x	2(x), 3x - x, 4x - 2x	x - 2x + 3x, x + 4x - 3x
3x	2x + x, 4x - x, 5x - 2x	3(x), 2(2x) - x, x + 4x - 2x
4x	2(2x), 3x + x, 5x - x	2(3x) - 2x, 2x - x + 3x
5x	3x + 2x, 4x + x, 6x - x	2(3x) - x, 2(2x) + x
6x	2(3x), 4x + 2x, 5x + x	3(2x), x + 2x + 3x

degree, and two or three partials enter into the generation of a third-degree combination tone.

There are many frequencies in the spectrum besides those shown in Table III, and there are many tones at each frequency. Since several tones sounding together at the same frequency form effectively just one tone, this duplication of the frequencies of partials can compensate to some extent for weak or missing partials. If the first partial were missing, its frequency could be supplied not by a primary tone but by the difference tones 3x - 2x and 2(2x)- 3x. This might help explain why the ear can hear the pitch associated with a complex tone even when the fundamental is weak or missing (Ch.1, end of Sec. C).

The aural spectrum generated by <u>two</u> complex tones sounding together is composed of the same frequencies as that generated by two pure tones of sufficient intensity; but, as in the spectrum of a single complex tone, tones of higher degree duplicate the frequencies of tones of lower degree. Let x be the frequency of the higher generating tone and y be that of the lower. Then the frequency of a spectral tone can always be represented in the form $mx + ny$, where m and n are integers one of which may be zero or negative. If $n = 0$, m is positive and the frequency is a multiple of x as in Table III. If $m = 0$, n is positive and the frequency is a multiple of y. If neither m nor n is 0, the tone is a combination tone and therefore of at least the second degree. The values of m and n thus divide the aural spectrum into three distinct parts: one that results from the action of the higher complex generator, one that results from the action of the lower, and one that results from the combined action of both generating tones.

The index of a tone is the numerical sum of m and n, which is written as $|m| + |n|$ and is not necessarily equal to the degree. For example, the generating tones of $2(2y) - x$ are x and 2y, its degree is 3, $m = -1$, $n = 4$, and its index is 5. Table IV shows a number of combination tones classified according to index and degree. All the tones in a line have the same frequency and, sounding together, are heard as one effective tone. The expressions for the frequencies of the third-degree tones reduce to second-degree expressions, that is, to expressions combining a multiple of x with a multiple of y.

Figure 4 shows the spectral tones with indexes less than or equal to 6 in a continuum of intervals extending from the unison thru increasingly larger intervals to one of infinite size. In this figure, the ordinates measure frequency; the abscissas, points along the continuum. The higher tone remains at the same frequency throughout; the lower tone starts at the same frequency as the higher tone and descends thru all intermediate frequencies to zero.

Table IV

Combination Tones Generated by the Joint Action of Two Complex Tones

The frequency of a partial is placed in parentheses when it is multiplied by a number other than 1. All the tones in a line have the same frequency.

Index	Degree	
	2	3
2	$x \pm y$	$2x - x \pm y$, $x \pm (2y - y)$
3	$2y \pm x$	$2(y) \pm x$, $3y - y \pm x$, $2y \pm (2x - x)$
	$2x \pm y$	$2(x) \pm y$, $3x - x \pm y$, $2x \pm (2y - y)$
4	$2x \pm 2y$	$2(x) \pm 2y$, $2x \pm 2(y)$, $3x - x \pm 2y$
	$3y \pm x$	$y + 2y \pm x$, $3y \pm (2x - x)$
	$3x \pm y$	$x + 2x \pm y$, $4x - x \pm y$
5	$3y \pm 2x$	$3y \pm 2(x)$, $y + 2y \pm 2x$
	$3x \pm 2y$	$3x \pm 2(y)$, $x + 2x \pm 2y$
	$4y \pm x$	$2(2y) \pm x$, $y + 3y \pm x$
	$4x \pm y$	$2(2x) \pm y$, $x + 3x \pm y$
6	$3x \pm 3y$	$x + 2x \pm 3y$, $3x \pm (y + 2y)$
	$4y \pm 2x$	$2(2y) \pm 2x$, $4y \pm 2(x)$, $y + 3y \pm 2x$
	$4x \pm 2y$	$2(2x) \pm 2y$, $4x \pm 2(y)$, $x + 3x \pm 2y$
	$5y \pm x$	$2y + 3y \pm x$, $y + 4y \pm x$
	$5x \pm y$	$2x + 3x \pm y$, $x + 4x \pm y$

Here we see the aural response to the acoustic stimulus pictured in Figure 3, Ch. 1, Sec. D; and, in passing from stimulus to response, we observe a remarkable increase in the number of spectral tones and in the number of intervals that attract attention by the coincidence of spectral tones. There were twelve such intervals; now there are twenty-three.

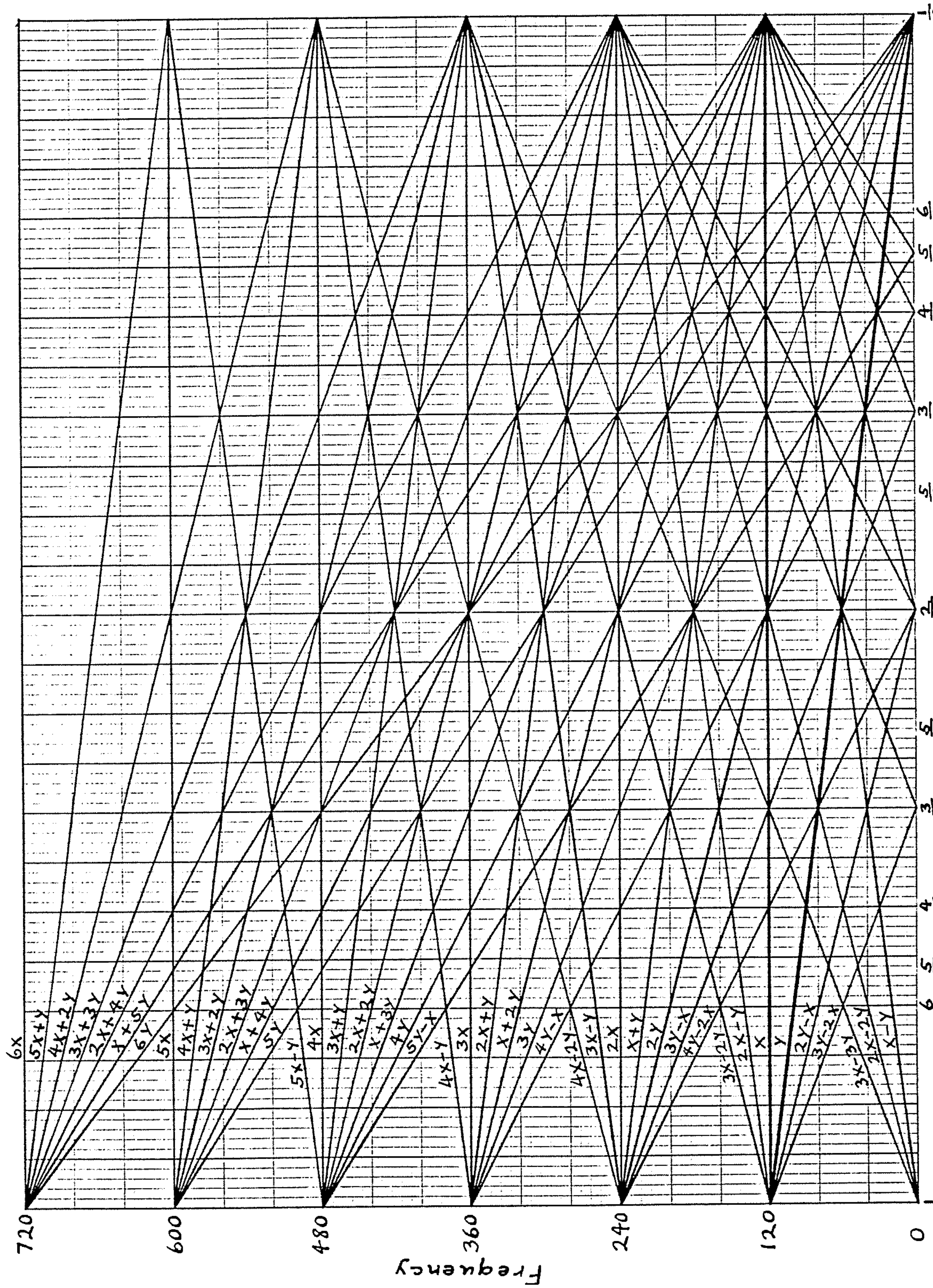


Figure 4. A portion of the aural spectrum of a continuum of intervals from the unison to one that is infinitely large.

The frequency ratios of the eleven new intervals are 7/5, 7/4, 7/3, 8/3, 7/2, 9/2, 7/1, 8/1, 9/1, 10/1, and 11/1. Of these, the ratios containing 7 in the numerator are not commonly recognized as being musically significant or desirable but perhaps should be.

Some difference tones vanish at certain ratios of x to y. In such a case, this means that $mx + ny = 0$ when $x/y = -n/m$, a commensurable ratio inasmuch as m and n are integers. Let x° and y° be the smallest positive integers that can express this ratio of x to y. Then $m/y^\circ = -n/x^\circ = k$, a $\pm$ integer. If $k = 1$, $m = y^\circ$, $n = -x^\circ$, and the frequency of the difference tone is $y^\circ x - x^\circ y > 0$ when $x/y > x^\circ/y^\circ$. If $k = -1$, $m = -y^\circ$, $n = x^\circ$, and the frequency of the difference tone is $x^\circ y - y^\circ x > 0$ when $x/y < x^\circ/y^\circ$. These two difference tones are called alternate difference tones here and each has a positive frequency in its proper range of x/y. This can be seen in Figure 4, where the tone $3y - 2x > 0$ when $x/y < 3/2$, vanishes when $x/y = 3/2$, and is replaced by its alternate $2x - 3y$ when $x/y > 3/2$.

F. Loudness Profile of the Typical Aural Spectrum

The typical aural spectrum is defined here as the aural spectrum generated by two typical musical tones of equal intensities sounding together, and the typical musical tone is characterized here as having a limited number of audible partials such that higher partials are weaker than lower ones. Let N be the number of audible partials; then N is also the number of the highest (and weakest) audible partial. The number of a partial of frequency rx is r, and the partial will be considered significant when $1 \leq r \leq N$ and negligible when $r > N$. Table I, Chapter 1, roughly indicates that, on the average, each partial is 6 decibels weaker than the one immediately preceding it in the harmonic series. Thus, if IL_r is the intensity level of partial r and IL_N is that of partial N,

$$IL_r = IL_N + 6(N - r) \qquad \text{F1}$$

Likewise, the number of a partial of frequency sy is s, and its intensity level is

$$IL_s = IL_N + 6(N - s) \qquad \text{F2}$$

To equate the intensities of the two generating tones, it is sufficient to specify that

$$IL_r = IL_s \text{ when } r = s. \qquad \text{F3}$$

Let the ear be presented with a harmonic interval sounded by two typical musical tones. Then assembling the aural responses to all the partials taken two at a time is sufficient to determine all the frequencies in the aural spectrum. We now adopt the hypothesis that assembling the contributions to the aural spectrum of all the partials taken two at a time is also sufficient to determine the approximate or probable intensity relations among all the tones of the aural spectrum. On this basis, the rules at the end of Section D regarding loudness and intensity relations in an aural spectrum generated by two pure tones of equal intensity can be used to determine the probable intensity relations in the typical aural spectrum. In this application, Rules 2 and 5 can be combined to the effect that, given different response tones of the same class and the same degree, those generated by stronger partials are louder than those generated by weaker partials. Furthermore, with the encouragement of Rules 1 and 3, we shall associate greater loudness with lower degree and disregard spectral tones of greater than the third degree.

The frequency of a primary tone or an aural harmonic generated by a partial of frequency rx equals

$$f = a(rx) = (ar)x \qquad \text{F4}$$

where $a = 1, 2,$ or 3 = the degree of the tone.
The index is $m = ar \geqq a$ F5

Inasmuch as the index multiplies the number of the partial by the degree of the tone, it relates directly to the loudness of the response tone in such a way as to be smaller when the tone is louder and larger when the tone is softer.

The frequency of a summation tone generated by partials of frequencies rx and sx $(r \neq s)$ equals

$$f = a(rx) + b(sx) = (ar + bs)x \qquad \text{F6}$$

where $a = 1$ or 2, $b = 1$ or 2, and $a + b = 2$ or 3 = the degree of the tone. The index is $m = ar + bs > a + b$ F7

and relates to the loudness of the tone in essentially the same way as in the case of the aural harmonic.

The frequency of a difference tone generated by partials rx and sx is $f = a(rx) - b(sx) = (ar - bs)x > 0$

$$\text{or } b(sx) - a(rx) = (bs - ar)x > 0 \qquad \text{F8}$$

where, as before, $a + b$ = the degree of the tone.
The index is $m = ar - bs < ar + bs$

$$\text{or } bs - ar < ar + bs \qquad \text{F9}$$

but, because of the minus sign, it does not relate to the loudness in the same way as in the previous case. However, by being smaller than the index of a summation tone of the same degree generated by the same partials, it does reflect the greater loudness of the difference tone (Rule 4).

The frequency of a combination tone generated by partials rx and sy is $f = a(rx) \pm b(sy) = (ar)x \pm (bs)y$

$$\text{or } b(sy) \pm a(rx) = (bs)y \pm (ar)x \qquad \text{F10}$$

The degree is $a + b$, $m = \pm ar$, $n = \pm bs$, and the index is

$$|m| + |n| = ar + bs \geqq a + b \qquad \text{F11}$$

Once again, the index relates to the loudness of the tone in such a way as to be smaller when the tone is louder and larger when the tone is softer.

When the generating tones are complex, different spectral tones may have equal frequencies whatever values are given to x and y. Such tones are said to have identical frequencies, or one is said to duplicate the frequency of the other. When the expressions for the frequencies of such tones are reduced to the form mx + ny, they all have the same values of m and n. It follows that spectral tones with identical frequencies have equal indexes and that the effective tone resulting from their union has their common index. Examples appear in Tables III and IV and are not to be confused with the coinciding tones of Figure 4, where different effective tones have equal frequencies only at certain ratios of x to y.

The intensity of an effective spectral tone is usually the sum of the intensities of the component simple tones, but its intensity level is practically that of the strongest component tone. Since the intensity level (or loudness) of this tone must be related to its index in the usual way, and since its index is identical to that of the effective tone, it follows that the loudness of an effective spectral tone is related to its index in the usual way. As a convenient approximation, then, we say that the loudness of an effective spectral tone generated by typical musical tones is usually greater for smaller indexes and less for greater indexes.

G. Masking

When two pure tones of approximately the same frequency but not so close as to produce beats are sounded together, the perceived loudness of one is affected by the presence of the other. If one of the tones is made louder than the other, the loudness of the other is diminished even tho its intensity remains the same. In fact, making one of the tones sufficiently loud will render the other completely inaudible in its presence. This phenomenon is called masking, and the tone whose loudness is diminished by the presence of the other is said to be masked by the other.

A pure tone of great enough intensity to generate aural harmonics masks another, weaker tone at frequencies not only near its own but also near those of its harmonics and somewhat less in the intervals between the harmonics. Thus, when two tones are widely separated in frequency, the lower will often mask the upper but the upper may not be able to mask the lower.

When two tones are brought closely enough together in frequency to create beats, one cannot mask the other because the hearing process no longer separates the two. It rather hears only one modulated tone, which is an equally valid physical interpretation of the fact as has been shown. One tone does not mask itself.

Generally speaking, the stronger partials fall at the lower frequencies in the spectrum of an interval composed of typical musical tones. This doubtless works in cooperation with the effects of masking to render some of the higher spectral tones inaudible in the sense that, even tho they may be physically present in the ear, the listener cannot directly detect their presence. It is impossible and unnecessary to draw an exact line of demarcation between the consciously audible and inaudible tones of the aural spectrum of an interval. It is enough to know that masking discriminates against the weaker and higher tones of the spectrum, thereby intensifying differences already found to exist therein.

Chapter 3

THE AURAL SPECTRA OF COMMENSURABLE INTERVALS

A. Fundamental Aspects

As in the preceding chapters, x and y are the respective frequencies of the higher and lower generating tones of an interval. Accordingly,

$$x \geqq y > 0 \qquad \text{A1}$$

A commensurable interval is an interval in which the ratio of x to y is commensurable, as explained in Chapter 1, Section D. When the interval is commensurable, x and y have a greatest common measure (or divisor), which is designated here by g. Letting $x°$ and $y°$ be the smallest positive integers that can express the ratio of x to y, we note that

$$x° = x/g = \text{a positive integer} \qquad \text{A2}$$

$$y° = y/g = \text{a positive integer} \qquad \text{A3}$$

$$x/y = x°/y° = \text{a rational number} \qquad \text{A4}$$

$$y°x = x°y \qquad \text{A5}$$

$$x° \geqq y° > 0 \qquad \text{A6}$$

Having 1 as their greatest common divisor, $x°$ and $y°$ are said to be relatively prime.

Let f be the frequency of any tone in the aural spectrum of an interval; then

$$f = mx + ny \qquad \text{A7}$$

where m and n are the coefficients of the tone, m being the coefficient of x and n being that of y. For example, in the tone 3x - y, m equals 3, and n equals -1; and, in the tone 5y, m equals 0, and n equals 5. Either m or n may be zero or negative, but not both; at least one coefficient must be positive in order for the spectral tone to have a positive frequency.

The frequency number of mx + ny is

$$f^\circ = mx^\circ + ny^\circ \qquad \text{A8}$$

whence

$$f = f^\circ g \qquad \text{A9}$$

The higher arithmetic teaches that, when x and y are given, integral values of m and n can always be found such that f can be any multiple of g and that there can be any number of spectral tones at every such frequency. Transforming A8 to

$$n = (f^\circ - mx^\circ)/y^\circ \qquad \text{A10}$$

shows that it is equivalent to a linear congruence to the modulus y° and can be solved for m and n by choosing from any y° consecutive integers the one value of m that makes $f^\circ - mx^\circ$ a multiple of y°. Since m can be chosen from any y° consecutive integers, there are infinitely many solutions for any given value of f°. Inasmuch as f can be any multiple of g, we recognize that the frequency number equals the number of distinct frequencies in the spectrum starting with g and ending with f.

An effective tone (often referred to simply as a tone) of the aural spectrum of an interval is completely identified when the coefficients m and n are each given a value, and different effective spectral tones are distinguished by different values of the coefficients. Because of this unique relationship, m and n are used as coordinates to plot the aural spectrum of a given interval.

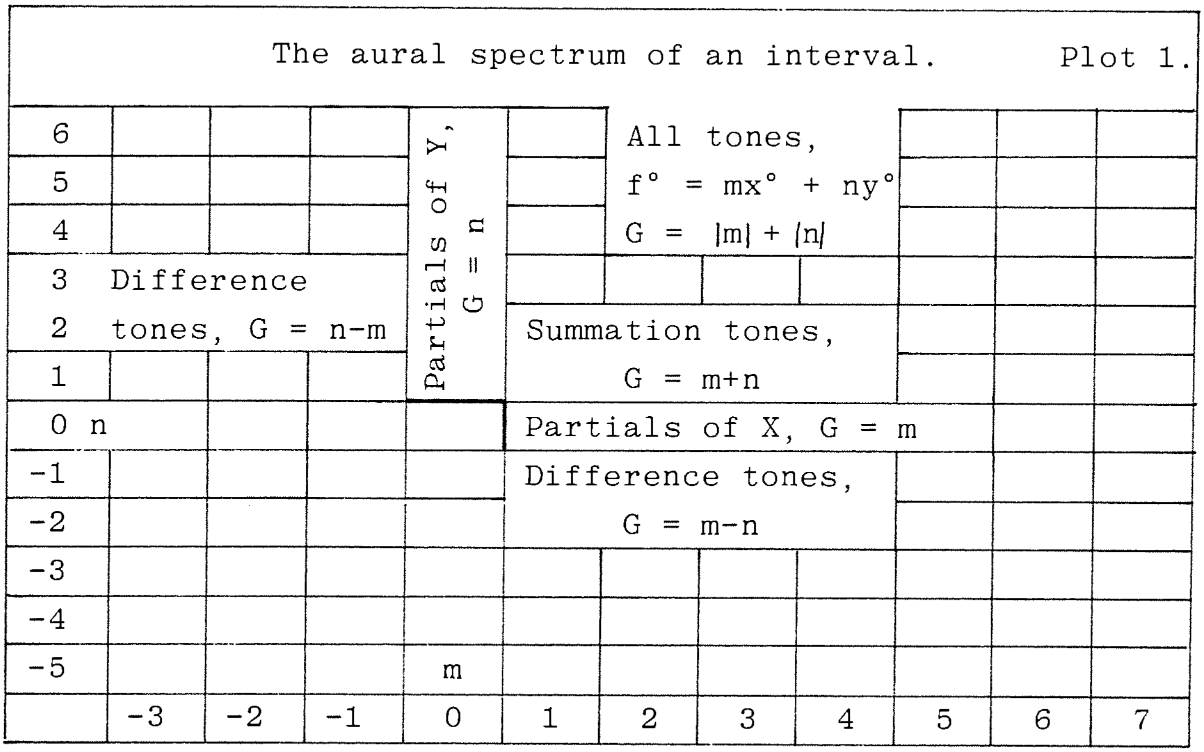
x and y determine the frequencies of the spectral tones; and, when the ratio of x to y is commensurable, they determine g, x°, y°, and the frequency numbers f°. Commensurable ratios are the object of investigation here, and actual frequencies are of little or no interest; therefore, each plot is identified with specific values of x° and y°

and displays values of f° even tho equations are expressed in terms of x, y, and f when possible in order to gain greater generality.

In order to plot the aural spectrum of an interval, the author empolys a network of lines that provides rows and columns of squares or rectangles in which to display the values of f° corresponding to the tones of the spectrum. Consecutive values of m and n are assigned respectively to the columns and rows of the network. Then each rectangle, standing at the intersection of a column and a row, has a value of m and a value of n; and f° is calculated for the rectangle by A8.

For the sake of convenient terminology, let the complex tone whose frequency is x be designated by capital X, let the tone whose frequency is y be designated by capital Y, and let the effective tones of the aural spectrum be named according to the spectral tones of lowest degree or the generating tones that have their frequencies. Thus, effective spectral tones whose frequencies are multiples of x are called partials of X, those whose frequencies are multiples of y are called partials of Y, and those whose frequencies combine a multiple of x with a multiple of y are called either summation tones or difference tones. As is shown in Plot 1, partials of X appear in the row $n = 0$, partials of Y appear in the column $m = 0$, summation tones appear in the first quadrant, and difference tones appear in the second and fourth quadrants. No tones lie in the third quadrant, where both m and n are zero or negative, because f° must be positive. Plot 2 presents a numerical example.

In order to simplify the mathematical descriptions, the quadrants are permitted to overlap, that is, to include the adjoining partials. Thus, the first quadrant embraces all the partials as well as the summation tones, the second



x° = 5, y° = 3

f° = 5m + 3n

Plot 2.

6											
5			10	15	20	25					
4		2	7	12	17	22	27				
3			4	9	14	19	24	29			
2			1	6	11	16	21	26	31		
1				3	8	13	18	23	28		
0 n					5	10	15	20	25		
-1					2	7	12	17			
-2						4	9				
-3						1					
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

quadrant takes in the partials of Y along with the difference tones in which m is negative, and the fourth quadrant includes the partials of X with the difference tones in which n is negative. No difficulty arises from the double classification of the partials, that is, the inclusion of the partials of Y in the first and second quadrants and the partials of X in the first and fourth quadrants. Five special expressions for the loudness index G are shown in Plot 1, but only three (for the first, second, and fourth quadrants) are needed when overlapping of the quadrants is permitted.

B. The Loudness Index

The loudness index (or simply index) of a spectral tone is the numerical sum of its coefficients. Let this be designated by G. Then the index of mx + ny is

$$G = |m| + |n| \qquad \text{B1}$$

As found in Section F, Chapter 2, the index is inversely indicative of the relative loudness, a larger index indicating a weaker tone, and a smaller index indicating a stronger tone. Thus, the index makes it easily possible to rank the spectral tones according to their relative loudness. Plot 3 shows all the spectral tones that have a given common index of 6.

The general formula B1 is sufficient for the direct calculation of G from given values of m and n, but the following special formulas for the different quadrants are needed to simplify expressions that interrelate f and G.

For all partials and summation tones, which are in the first quadrant, neither m nor n is negative and

$$G = m + n \qquad \text{B2}$$

Eliminating first n and then m between A7 and B2 results in

$x° = 7,\ y° = 5,\ G = 6$ Plot 3.

7											
6				30							
5			18		32						
4		6				34					
3							36				
2								38			
1									40		
0 n										42	
-1									30		
-2								18			
-3							6				
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

$$f = m(x - y) + Gy \quad \text{B3}$$
$$= Gx - n(x - y) \quad \text{B4}$$

and, since $x - y \geqq 0$,

$$Gy \leqq f \leqq Gx \quad \text{B5}$$

From this, we draw the rule that the summation tones for given values of x, y, and G lie between the partials Gy and Gx.

For partials of Y and difference tones in the second quadrant, $m \leqq 0$ and

$$G = n - m \quad \text{B6}$$

Combining this with A7 results in

$$f = m(x + y) + Gy \quad \text{B7}$$
$$= n(x + y) - Gx \quad \text{B8}$$

Since $f > 0$, these equations lead to the following inequalities:

$$-Gy/(x + y) < m \leqq 0 \quad \text{B9}$$

$$Gx/(x + y) < n \leqq G \quad \text{B10}$$

For partials of X and difference tones in the fourth quadrant, $n \leqq 0$ and

$$G = m - n \quad \text{B11}$$

Taking this together with A7 results in

$$f = m(x + y) - Gy \quad \text{B12}$$

$$= n(x + y) + Gx \quad \text{B13}$$

and, since $f > 0$,

$$Gy/(x + y) < m \leqq G \quad \text{B14}$$

$$-Gx/(x + y) < n \leqq 0 \quad \text{B15}$$

Let us consider the frequency relations that exist when G is constant. In the first quadrant, when m increases by 1, n decreases by 1, and f increases by $x - y$. In the second and fourth quadrants, when m increases by 1, n increases by 1, and f increases by $x + y$. Thus, when G is constant, f always increases when m increases, and, since G is the maximum value of m, the partial Gx is the highest spectral tone with a given index G.

It is equally true that f decreases when m decreases; but there are two minimum values of m, hence f, one in the second quadrant and the other in the fourth. In both of these quadrants, f decreases by $x + y$ when m decreases by 1; but the least value of m is determined by the requirement that f be greater than 0; therefore, both minimum frequencies with G constant conform to the rule that

$$0 < f \leqq x + y \quad \text{B16}$$

The frequency range thus established, which incidentally includes the lowest summation tone, is referred to here as the basic range of the aural spectrum, and the tones in this range may be called minimum-frequency tones. See Section I.

C. Pairs of Spectral Tones

As given in Section A, m and n are the coefficients of any tone in the aural spectrum of an interval, f is its frequency, and f° is its frequency number. Let m' and n' be the coefficients of another tone of the spectrum, let f' be its frequency, and let f'° be its frequency number. Then

$$f' = m'x + n'y \qquad C1$$

and

$$f'^\circ = m'x^\circ + n'y^\circ \qquad C2$$

Subtraction of C1 from A7 gives the difference between the frequencies of the two tones:

$$f - f' = (m - m')x - (n' - n)y \qquad C3$$

and subtraction of C2 from A8 gives the difference between their frequency numbers:

$$f^\circ - f'^\circ = (m - m')x^\circ - (n' - n)y^\circ \qquad C4$$

C4 transforms to

$$n' - n = [(m - m')x^\circ - (f^\circ - f'^\circ)]/y^\circ \qquad C5$$

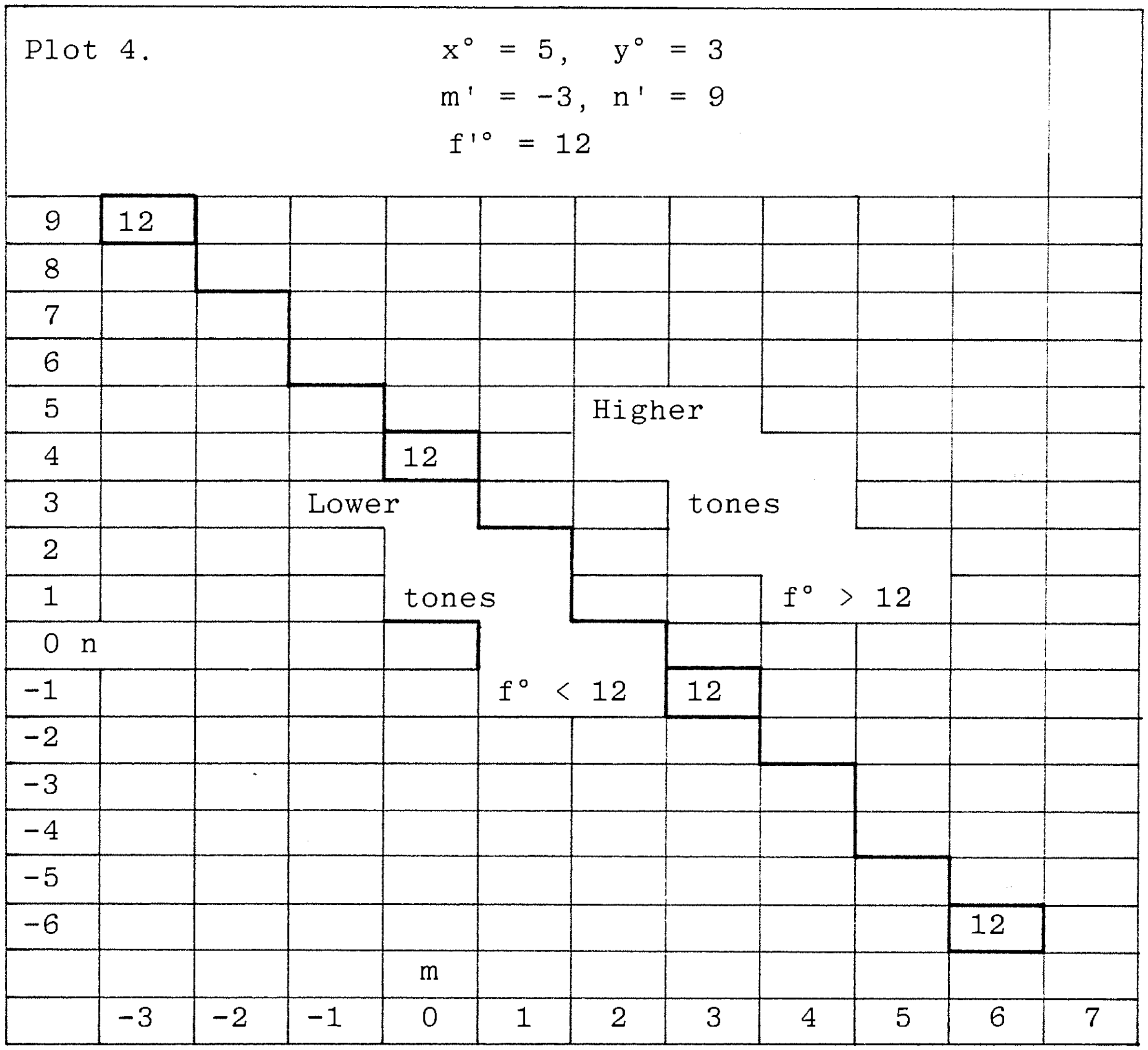
which can always be solved for integral values of m - m' and n' - n whatever integer $f^\circ - f'^\circ$ may be.

$$\text{If } f^\circ > f'^\circ,\ n' - n < (m - m')x^\circ/y^\circ \qquad C6$$

$$\text{If } f^\circ < f'^\circ,\ n' - n > (m - m')x^\circ/y^\circ \qquad C7$$

$$\text{If } f^\circ = f'^\circ,\ n' - n = (m - m')x^\circ/y^\circ \qquad C8$$

These relations are illustrated in Plot 4, where the tones 9y - 3x, 4y, 3x - y, and 6x - 6y all have a frequency number of 12, and the lines joining these tones separate the tones that are lower than these from those that are higher. It is



helpful to compare this plot with Plot 2, which has the same ratio of x to y.

When the tones coincide, $f° = f'°$ and it is clear from C8 and the relative primality of $x°$ and $y°$ that $m - m'$ must be a multiple of $y°$. Let this multiple be represented by k; then

$$m - m' = ky° \qquad \text{C9}$$

$$n' - n = kx° \qquad \text{C10}$$

and it is seen that k is the greatest common divisor of

$m - m'$ and $n' - n$. When the coefficients of two tones in the spectrum of an interval satisfy C9 and C10, then, by C3 and A5, $f - f' = k(y°x - x°y) = 0$, and their frequencies are the same. Thus, $mx + ny = m'x + n'y$ when, and only when, C9 and C10 are satisfied.

k can be any integer, but we are not interested in the trivial case that results when $k = 0$, and we observe that the distinction between positive and negative values of k only involves the question as to which of two tones is identified as $mx + ny$ while the other is designated as $m'x + n'y$. Because of this, we may and usually do limit k to being positive. The coinciding tones are said to be adjacent when $k = 1$.

Recalling A6, C9, and C10, and letting k be positive, we get

$$n' - n \geqq m - m' > 0 \qquad \text{C11}$$

whence it follows that

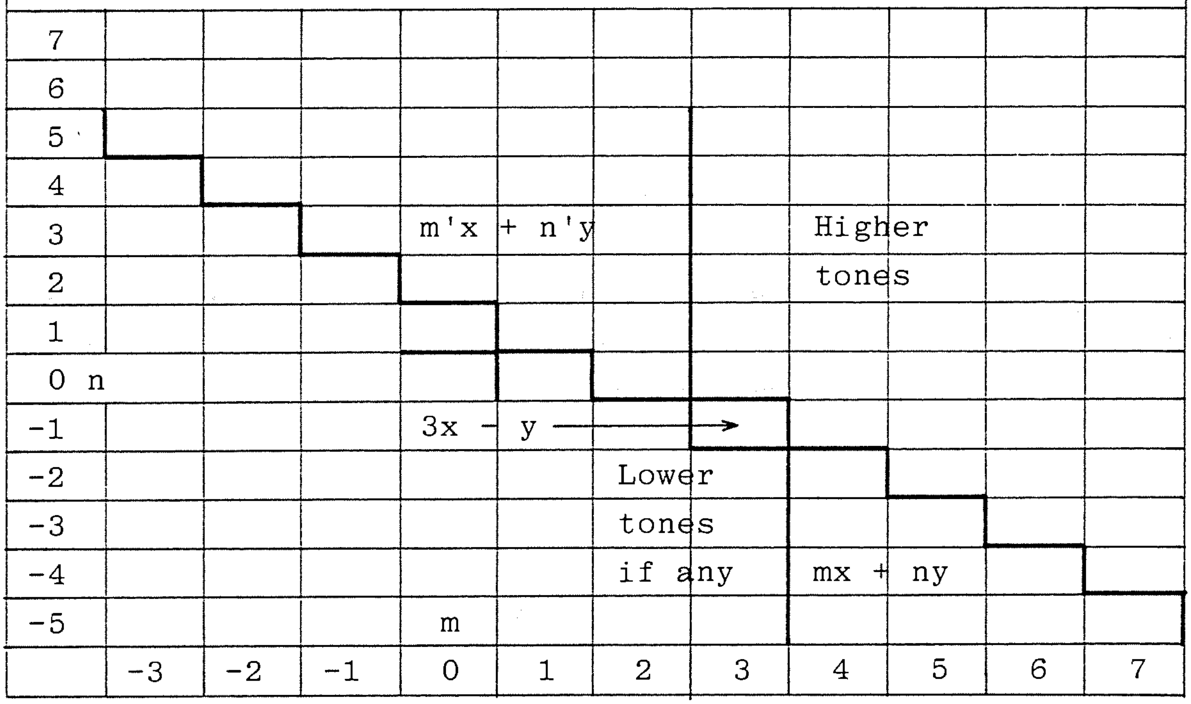
$$m' + n' \geqq m + n \qquad \text{C12}$$

$$m > m' \text{ and } n' > n \qquad \text{C13}$$

Plot 5 illustrates the significance of these relations for all intervals that satisfy A1 by showing the areas where all tones of the same frequency as $3x - y$ must lie. If this tone is interpreted as $mx + ny$, then other tones of the same frequency lie in the area identified with $m'x + n'y$, where $m' < 3$ and $m' + n' \geqq 2$. If $3x - y$ is interpreted as $m'x + n'y$, then other tones of the same frequency lie in the area identified with $mx + ny$, where $m > 3$ and $m + n \leqq 2$. Attempting to equate the frequency of a tone outside both of these areas to $3x - y$ results in a violation of A1.

C9 and C10 can be used to determine the ratio of x to y at which two given tones will coincide. As an example, let us take the two tones presented in the second paragraph of

Plot 5. Significance of C12 and C13



Section A. When they coincide, 3x - y = 5y, whence, respecting C12 and C13, m' = 0, n' = 5, m = 3, n = -1

$$n' - n = 6, \quad m - m' = 3, \quad k = 3$$

$$x^\circ = 2, \quad y^\circ = 1$$

and, by A4, $$x/y = 2/1$$

which is the only ratio of x to y at which 3x - y will coincide with 5y. This coincidence can be seen at the frequency of 300 in Figure 4, Chapter 2.

Furthermore, when x°, y°, and k are given, C9 can easily be solved for m' as a function of m or for m as a function of m', and C10 can be solved for n' as a function of n or for n as a function of n'. By varying k, then, we can also use C9 and C10 to find tones in the spectrum of a given interval that coincide with a particular spectral tone. This is exemplified in Plot 4, which shows tones that coincide with $9y - 3x$ when $x/y = 5/3$.

When the coinciding tones are assigned to specific quadrants in conformity to C12 and C13, then six cases collectively containing all possible coincidences arise. These are identified by two-digit case numbers, the first digit indicating the quadrant occupied by $m'x + n'y$, and the second digit indicating the quadrant occupied by $mx + ny$. For example, Case 21 places $m'x + n'y$ in the second quadrant and $mx + ny$ in the first. The six cases are 21, 14, 11, 22, 44, and 24. No others are needed to account for any coincidences.

D. The Indexes of Coinciding Spectral Tones

The loudness index of $mx + ny$ is defined by B1; that of $m'x + n'y$ is

$$G' = |n'| + |m'| \qquad \text{D1}$$

The forms that G takes in the different quadrants are presented in B2, B6, and B11. The forms that G' and G take in the six cases of coinciding tones are tabulated below:

Case	G'	G
21	$n' - m'$	$m + n$
14	$n' + m'$	$m - n$
11	$n' + m'$	$m + n$
22	$n' - m'$	$n - m$
44	$m' - n'$	$m - n$
24	$n' - m'$	$m - n$

D2

Adding C9 to C10 results in

$$k(x° + y°) = n' - m' + m - n \qquad \text{D3}$$

subtracting C9 from C10 gives

$$k(x° - y°) = n' + m' - m - n \qquad \text{D4}$$

and comparing these with the preceding expressions for G' and G discloses the following relationships:

Case 21. $G' - G + 2m' = k(x^\circ - y^\circ)$ D5

$G' - G + 2m = k(x^\circ + y^\circ) = G' + G - 2n$ D6

Case 14. $G' - G - 2n = k(x^\circ - y^\circ)$ D7

$G - G' + 2n' = k(x^\circ + y^\circ) = G' + G - 2m'$ D8

Case 11. $G' - G = k(x^\circ - y^\circ)$ D9

$G' - G + 2(m - m') = k(x^\circ + y^\circ) = G' + G - 2(m' + n)$ D10

Case 22. $G' - G = k(x^\circ + y^\circ)$ D11

Case 44. $G - G' = k(x^\circ + y^\circ)$ D12

Case 24. $k(x^\circ + y^\circ) = G' + G$ D13

In all cases, the quantities k, x°, y°, G, and G' conform to the rule that $|G' - G| \leqq k(x^\circ + y^\circ) \leqq G' + G$ D14

where $k(x^\circ + y^\circ)$ must differ from $|G' - G|$ or $G' + G$ or both by an even number; and any set of these quantities that conforms to this rule can be identified as to case and used in solving for one or more corresponding sets of coefficients of coinciding spectral tones.

<u>Case 21.</u> Inasmuch as $m' \leqq 0$, D5 shows that

$$G' - G \geqq k(x^\circ - y^\circ) \quad \text{D15}$$

which identifies the case and, since $x^\circ \geqq y^\circ$, indicates that $G' - G = |G' - G|$. Because of this and the fact that m and n are greater than or equal to zero, D6 demonstrates conformity to D14. Furthermore, D6 determines m and n uniquely, and m' and n' can be obtained from these by means of C9 and C10. This assures that the two tones will have the same frequency, which can only be positive because mx + ny is in the first quadrant.

<u>Case 14.</u> Since $n \leqq 0$, D7 shows that

$$G' - G \leqq k(x^\circ - y^\circ) \quad \text{D16}$$

which contrasts with D15 and distinguishes Case 14 from Case 21 as long as the related quantities are not equal as in D9. Inasmuch as m' and n' are greater than or equal to 0, D8 and D16 together exhibit conformity to D14. Furthermore, D8 can be solved for m' and n', and these used in C9 and C10 determine m and n so that $mx + ny = m'x + n'y > 0$.

<u>Case 11.</u> D9 identifies this case and shows that $G' - G = |G' - G|$. This together with C11, D10, and the fact that m' and n are not negative assures us of conformity to D14. In this case, D10 does not uniquely determine a pair of coefficients, but it does give

$$m' + n = [G' + G - k(x^\circ + y^\circ)]/2 \qquad \text{D17}$$

which, by D9, simplifies to

$$m' + n = G' - kx^\circ = G - ky^\circ \qquad \text{D18}$$

Since $m' \geqq 0$ and $n \geqq 0$, it follows that m' can be any integer that satisfies

$$0 \leqq m' \leqq G' - kx^\circ \qquad \text{D19}$$

and

$$n = G' - kx^\circ - m' \qquad \text{D20}$$

C9 and C10 then use m' and n to determine m and n' such that the two tones have the same positive frequency.

<u>Cases 22, 44, and 24.</u> Equations D11, D12, and D13 identify their respective cases and conform to D14, but they do not determine coefficients. Therefore, a solution for a set of coefficients must be initiated by choosing a coefficient. Then C9, C10, and the appropriate equations in D2 may be used to solve for the three remaining coefficients of the set. The choice of a coefficient cannot be purely arbitrary, however. In order to ensure positive frequencies and conformity to the limitations of magnitude imposed by the indexes, B9, B10, B14, B15, and their following application to $m'x + n'y$ must be satisfied by the coefficients that are

to be determined.

When $m'x + n'y$ is in the second quadrant, B9 and B10 give

$$-G'y/(x + y) < m' \leqq 0 \quad \text{D21}$$

$$G'x/(x + y) < n' \leqq G' \quad \text{D22}$$

When $m'x + n'y$ is in the fourth quadrant, B14 and B15 give

$$G'y/(x + y) < m' \leqq G' \quad \text{D23}$$

$$-G'x/(x + y) < n' \leqq 0 \quad \text{D24}$$

B9 and B10 are consistent with B6 in that, if m and n are related as in B6, and if m satisfies B9, then n satisfies B10, or if n satisfies B10, then m satisfies B9. In the same way, B14 and B15 are consistent with B11, D21 and D22 are consistent with $G' = n' - m'$, and D23 and D24 are consistent with $G' = m' - n'$.

In Case 22, it is necessary to satisfy B9, B10, D21, and D22; and, to do this, it is sufficient to satisfy B9 or B10. Substituting $m' + ky°$ for m in B9 and simplifying with the aid of D11 results in

$$-G'y/(x + y) < m' \leqq -ky°$$

which satisfies D21. Substituting $n' - kx°$ for n in B10 and simplifying results in

$$G'x/(x + y) < n' \leqq G' - ky°$$

which satisfies D22.

In Case 44, it is sufficient to satisfy D23 or D24. Substituting $m - ky°$ for m' in D23 and simplifying with the aid of D12 results in

$$Gy/(x + y) < m \leqq G - kx°$$

which satisfies B14. Substituting $n + kx°$ for n' in D24 results in

$$-Gx/(x + y) < n \leqq -kx°$$

which satisfies B15.

In Case 24, it is necessary to satisfy B14, B15, D21, and D22. If $G \leqq ky°$, then by D13, $G' \geqq kx°$ and it is sufficient to satisfy B14 or B15. Substituting $m' + ky°$ for m in B14 and simplifying the result with the aid of D13 gives

$$-G'y/(x + y) < m' \leqq kx° - G'$$

which satisfies D21. Substituting $n' - kx°$ for n in B15 and simplifying with the aid of D13 gives

$$G'x/(x + y) < n' \leqq kx°$$

which satisfies D22. If $G \geqq ky°$, then by D13, $G' \leqq kx°$ and it is sufficient to satisfy D21 or D22. Substituting $m - ky°$ for m' in D21 and simplifying gives

$$Gy/(x + y) < m \leqq ky°$$

which satisfies B14. Substituting $n + kx°$ for n' in D22 and simplifying the result gives

$$-Gx/(x + y) < n \leqq ky° - G$$

which satisfies B15.

Three examples. In our first example, let it be given that $G' = 5$ and $G = 6$. Then $G' - G = -1$ and $G' + G = 11$; and, to satisfy D14, $k(x° + y°)$ must be an odd number such that $1 \leqq k(x° + y°) \leqq 11$. We choose $x° = 4$, $y° = 3$, and $k = 1$. Thus $k(x° + y°) = 7$, and $G' - G < k(x° + y°)$, which satisfies D16 and thereby identifies the case as 14. D8 gives

$$m' = [G' + G - k(x° + y°)]/2 = 2$$

$$n' = [k(x° + y°) + G' - G]/2 = 3$$

C9 gives $m = m' + ky° = 5$

C10 gives $n = n' - kx° = -1$

C2 and A8 give $f'° = f° = 17.$

Plot 6 shows these coinciding tones, plus some other tones

Plot 6. $x^\circ = 4$, $y^\circ = 3$, $k = 1$, Case 14

7											
6											
5				15							
4			8		16						
3		1	(G	' = 5	)	**[17]**	21				
2								22			
1									23		
0 n				**[ ]**			(	G = 6	)	24	
-1									**[17]**		
-2								10			
-3							3				
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

that have the same values of G' and G but do not coincide, in order not only to illustrate the above solution but also to demonstrate the use of a plot to find whatever coincidences may exist or to show the lack of coincidences if there are none.

In our second example, $G' = 6$, $G = 4$, $x^\circ = 3$, and $y^\circ = 2$. From this data, $G' - G = 2$, $G' + G = 10$, $x^\circ + y^\circ = 5$, and by D14, $2 \leqq 5k \leqq 10$, whence it is evident that $k = 2$, not 1, because $5k - 2$ and $10 - 5k$ must be even numbers.

$$k(x^\circ - y^\circ) = 2 = G' - G$$

and

$$k(x^\circ + y^\circ) = 10 = G' + G$$

thereby satisfying D15, D16, D9, and D13, which identify Cases 21, 14, 11, and 24. Because of this, D5 thru D10 may

$x^\circ = 3$, $y^\circ = 2$, $k = 2$, Case 24										Plot 7.
7											
6				12							
5			7		13						
4		2	(G' = 6)			14					
3											
2						10					
1							11				
0 n						(G = 4)		12			
-1							7				
-2						2					
-3											
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

be used to solve for the coefficients, whence m' = 0, n' = 6, m = 4, n = 0, and $f'^\circ = f^\circ = 12$. Regarded as Case 24, however, this data gives three solutions including the preceding. $G'y/(x + y) = 12/5$, and substitution into D21 yields

$-12/5 < m' \leq 0$, whence m' = -2, -1, or 0.
From D2, $n' = m' + G' = 4, 5$, or 6.
From C9, $m = m' + ky^\circ = 2, 3$, or 4.
From C10, $n = n' - kx^\circ = -2, -1$, or 0.
From C2 and A8, $f'^\circ = f^\circ = 2, 7$, or 12.

These and some other tones with the same indexes are shown in Plot 7.

Our third example shows what happens when the rule of D14 is violated. Let G' = 6, G = 4, $x^\circ = 7$, and $y^\circ = 5$. Then G' - G = 2, G' + G = 10, and $x^\circ + y^\circ = 12$. There is no value assignable to k that satisfies D14, the data do not

Plot 8. $x^\circ = 7$, $y^\circ = 5$, No coincidence.

7											
6				30							
5			18		32						
4		6		(G' = 6)		34					
3											
2						24					
1							26				
0 n						(G = 4)		28			
-1							16				
-2						4					
-3											
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

fit any of the prescribed cases, and none of the equations of D2 or of D5 thru D10 are usable to solve for a set of coefficients. As can be seen in Plot 8, the result is that there is no coincidence.

E. Minimum-Index Tones

Dividing B3, B7, and B12 by g and solving for G results respectively in the three following expressions for G as a function of f° and m:

In the first quadrant, $G = [f^\circ - m(x^\circ - y^\circ)]/y^\circ$ E1

In the second quadrant, $G = [f^\circ - m(x^\circ + y^\circ)]/y^\circ$ E2

In the fourth quadrant, $G = [m(x^\circ + y^\circ) - f^\circ]/y^\circ$ E3

A10 expresses n as a function of f° and m in one equation for all quadrants.

Our concern here is the behavior of G and n while f° is held constant and m is given successively larger values. As m increases, G decreases in the first and second quadrants and increases in the fourth quadrant whereas n decreases in every quadrant. Inasmuch as $n \geqq 0$ in the first and second quadrants and $n \leqq 0$ in the fourth, it is seen that, as m increases, the index G decreases when $n > 0$ and increases when $n \leqq 0$, reaching a minimum when n is near or equal to 0. Plots 9, 10, 11, and 12 illustrate this.

The tone with the smallest index of all tones of its frequency is called a minimum-index tone. It is the most significant tone of its frequency, and there is usually one such tone for every frequency as in Plots 9, 11, and 12; but there may be two for some frequencies as in Plot 10. In locating a minimum-index tone, there are always two adjacent tones to choose from: one having a least index for $n > 0$, the other having a least index for $n \leqq 0$. If the two indexes are unequal, the lesser is the minimum; if they are equal, both are a minimum. Let the tone in the first or second quadrant be $m'x + n'y$, and let the tone in the fourth quadrant be $mx + ny$; then Cases 14 and 24 are applicable with $k = 1$ because the tones are adjacent.

When the index of a tone in the first quadrant is to be compared with that of an adjacent tone in the fourth quadrant (Case 14), D8 and D7 give

$$G' + G = 2m' + x° + y° \qquad \text{E4}$$

and

$$G' - G = 2n' - (x° + y°) \qquad \text{E5}$$

$$= 2n + x° - y° \qquad \text{E6}$$

If $G' - G \geqq 0$,

$$n' \geqq (x° + y°)/2 \qquad \text{E7}$$

Plot 9. $x° = 3,\ y° = 2,\ f° = 1$

7											
6											
5	1 (G = 8)										
4											
3											
2			1 (G = 3)								
1											
0 n											
-1					1 (G = 2)						
-2											
-3											
-4							1 (G = 7)				
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

Plot 10. $x° = 3,\ y° = 1,\ f° = 2$

7											
6											
5			2 (G = 6)								
4											
3											
2				2 (G = 2)							
1											
0 n											
-1					2 (G = 2)						
-2											
-3											
-4						2 (G = 6)					
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

$x° = 3,\ y° = 2,\ f° = 7$ Plot 11.

7											
6											
5			7 (G = 6)								
4											
3											
2					7 (G = 3)						
1											
0 n											
-1							7 (G = 4)				
-2											
-3											
-4									7 (G = 9)		
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

$x° = 3,\ y° = 2,\ f° = 12$ Plot 12.

7											
6				12 (G = 6)							
5											
4											
3						12 (G = 5)					
2											
1											
0 n								12 (G = 4)			
-1											
-2											
-3										12 (G = 9)	
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

$$n \geqq -(x^\circ - y^\circ)/2 \tag{E8}$$

$$G' \geqq m' + (x^\circ + y^\circ)/2 \tag{E9}$$

$$G \leqq m' + (x^\circ + y^\circ)/2 = m + (x^\circ - y^\circ)/2 \tag{E10}$$

and G is a minimum index.

If $G' - G \leqq 0$,

$$n' \leqq (x^\circ + y^\circ)/2 \tag{E11}$$

$$n \leqq -(x^\circ - y^\circ)/2 \tag{E12}$$

$$G' \leqq m' + (x^\circ + y^\circ)/2 \tag{E13}$$

$$G \geqq m' + (x^\circ + y^\circ)/2 = m + (x^\circ - y^\circ)/2 \tag{E14}$$

and G' is a minimum index.

When the index of a tone in the second quadrant is to be compared with that of an adjacent tone in the fourth quadrant (Case 24), D13 gives

$$G' + G = x^\circ + y^\circ \tag{E15}$$

and

$$G' - G = x^\circ + y^\circ - 2G \tag{E16}$$

$$= 2G' - (x^\circ + y^\circ) \tag{E17}$$

If $G' - G \geqq 0$,

$$G' \geqq (x^\circ + y^\circ)/2 \tag{E18}$$

$$G \leqq (x^\circ + y^\circ)/2 \tag{E19}$$

and G is a minimum index.

If $G' - G \leqq 0$,

$$G' \leqq (x^\circ + y^\circ)/2 \tag{E20}$$

$$G \geqq (x^\circ + y^\circ)/2 \tag{E21}$$

and G' is a minimum index.

If $x^\circ \pm y^\circ$ is even, then, regardless of whether it is a tone in the first quadrant or in the second quadrant that coincides with a tone in the fourth quadrant, E5 and E16 show that G' and G may be the same; and if, when $x^\circ \pm y^\circ$ is even, G' and G are the same, then both indexes are a minimum, and there is a coincidence of minimum-index tones. If $x^\circ \pm y^\circ$ is odd, then G' and G must be different, only one of them is a minimum, and there cannot be a coincidence of minimum-index tones.

Any index that is less than or equal to $(x^\circ + y^\circ)/2$ satisfies the four equations E10, E13, E19, and E20 and is therefore a minimum. An index greater than $(x^\circ + y^\circ)/2$ that has $m' > 0$ (or $m > y^\circ$ by C9) and whose coefficient of y lies between $-(x^\circ - y^\circ)/2$ and $(x^\circ + y^\circ)/2$ inclusive[1] satisfies E10 or E13 and is a minimum. The boundaries thus established give the minimum-index tones a definite location in the plot of an aural spectrum. Tones that lie outside these boundaries have larger than minimum indexes and are called peripheral tones. Plot 13 shows the minimum-index tones with $G \leqq (x^\circ + y^\circ)/2$ for the interval 7/5, and Plot 14 shows minimum-index tones with $G > (x^\circ + y^\circ)/2$ for the same interval. Plot 15 shows minimum-index tones for the interval 8/3. Coincidences are present in Plots 13 and 14, where $x^\circ \pm y^\circ$ is even, and absent from Plot 15, where $x^\circ \pm y^\circ$ is odd. These plots do not show peripheral tones.

There is at least one minimum-index tone for every frequency that can exist in the spectrum, but all multiples of g are possible; therefore, the complete set of minimum-index tones includes all multiples of g. Thus, the frequency numbers in these plots start with 1 and, including all integers in succession, increase (that is, extend to the right) without limit. For this reason, the minimum-index tones are said to have perfect continuity and an unlimited range. The possession of these two attributes depends on the inclusion

1. The coefficient of y is either n or n'. If it is n, E8 is applicable; if it is n', E11 applies.

Plot 13. $x^\circ = 7$, $y^\circ = 5$, $G \leqq (x^\circ + y^\circ)/2$
Minimum-index tones.

7											
6				30							
5			18	25	32						
4		6	13	20	27	34					
3		1	8	15	22	29	36				
2			3	10	17	24	31	38			
1				5	12	19	26	33	40		
0 n					7	14	21	28	35	42	
-1					2	9	16	23	30		
-2						4	11	18			
-3							6				
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

Plot 14. $x^\circ = 7$, $y^\circ = 5$, $G > (x^\circ + y^\circ)/2$
Minimum-index tones.

7											
6					37	44	51	58	65	72	79
5						39	46	53	60	67	74
4							41	48	55	62	69
3								43	50	57	64
2									45	52	59
1										47	54
0 n											49
-1										37	44
-2											
-3											
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

x° = 8, y° = 3, Minimum-index tones. Plot 15.											
7											
6											
5				15	23	31	39	47	55		
4			4	12	20	28	36	44	52		
3			1	9	17	25	33	41	49		
2				6	14	22	30	38	46		
1				3	11	19	27	35	43		
0 n					8	16	24	32	40		
-1					5	13	21	29	37		
-2					2	10	18	26	34		
-3						7					
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

of indexes of all magnitudes, no matter how large. In the practical realm, however, indexes that are too great (that is, tones that are too weak) must be excluded from consideration. This subject is taken up in the next section.

F. Significant Tones of the Aural Spectrum

Let N denote the maximum index of an audibly significant spectral tone. Then tones whose indexes are less than or equal to N will be considered significant, and tones whose indexes are greater than N will be regarded as negligible. It is suggested here that 7 or 8 may be a suitable value of N for a typical spectrum, but this number is not known with certainty and may vary according to circumstances. Furthermore, it is advantageous for the development of principles to be able to vary the number of tones that are

Plot 16. $x° = 8,\ y° = 3,\ N = 6$

7											
6				18							
5			7	15	23	31	39	47			
4			4	12	20	28	36	44			
3			1	9	17	25	33	41	49		
2				6	14	22	30	38	46	54	
1				3	11	19	27	35	43	51	
0 n					8	16	24	32	40	48	
-1					5	13	21	29	37	45	
-2					2	10	18	26	34	42	
-3						7	15				
-4						4					
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

Plot 17. $x° = 7,\ y° = 3,\ N = 6$

7											
6				18							
5			8	15	22	29	36	43			
4			5	12	19	26	33	40			
3			2	9	16	23	30	37	44		
2				6	13	20	27	34	41	48	
1				3	10	17	24	31	38	45	
0 n					7	14	21	28	35	42	
-1					4	11	18	25	32	39	
-2					1	8	15	22	29	36	
-3						5	12				
-4						2					
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

considered significant; therefore, N is not limited here to one fixed value.

Assigning a finite value to N establishes a boundary as shown in Plots 16 and 17. The tones inside this boundary are recognized as being audibly significant; those outside the boundary are regarded as negligible. This boundary always divides the minimum-index tones into two categories: the significant and the negligible; and, when $N > (x^\circ + y^\circ)/2$, the peripheral tones are divided into the same two categories. Thus, there are four categories altogether, but only three of them are important, and they occur in three cases that collectively contain all possible values of $x^\circ + y^\circ$. These cases are as follows:

Case 1, Plot 18.

$$x^\circ + y^\circ = 2N \text{ or } 2N + 1 \qquad \text{F1}$$

The significant tones are identical to the minimum-index tones with $G \leqq (x^\circ + y^\circ)/2$.

Case 2, Plot 19.

$$x^\circ + y^\circ < 2N \qquad \text{F2}$$

Significant peripheral tones occur in this case.

Case 3, Plot 20.

$$x^\circ + y^\circ > 2N + 1 \qquad \text{F3}$$

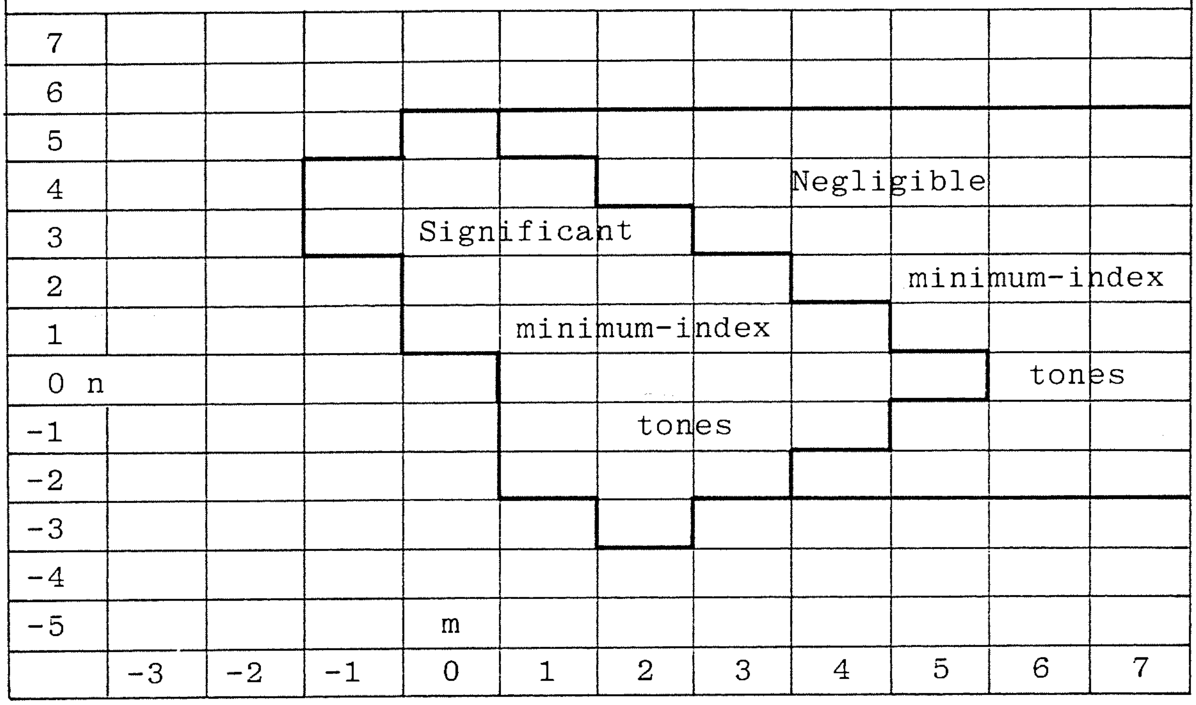
The significant tones do not include all of the minimum-index tones with $G \leqq (x^\circ + y^\circ)/2$.

We are not concerned with negligible peripheral tones. The importance of certain negligible minimum-index tones becomes apparent in the following paragraphs.

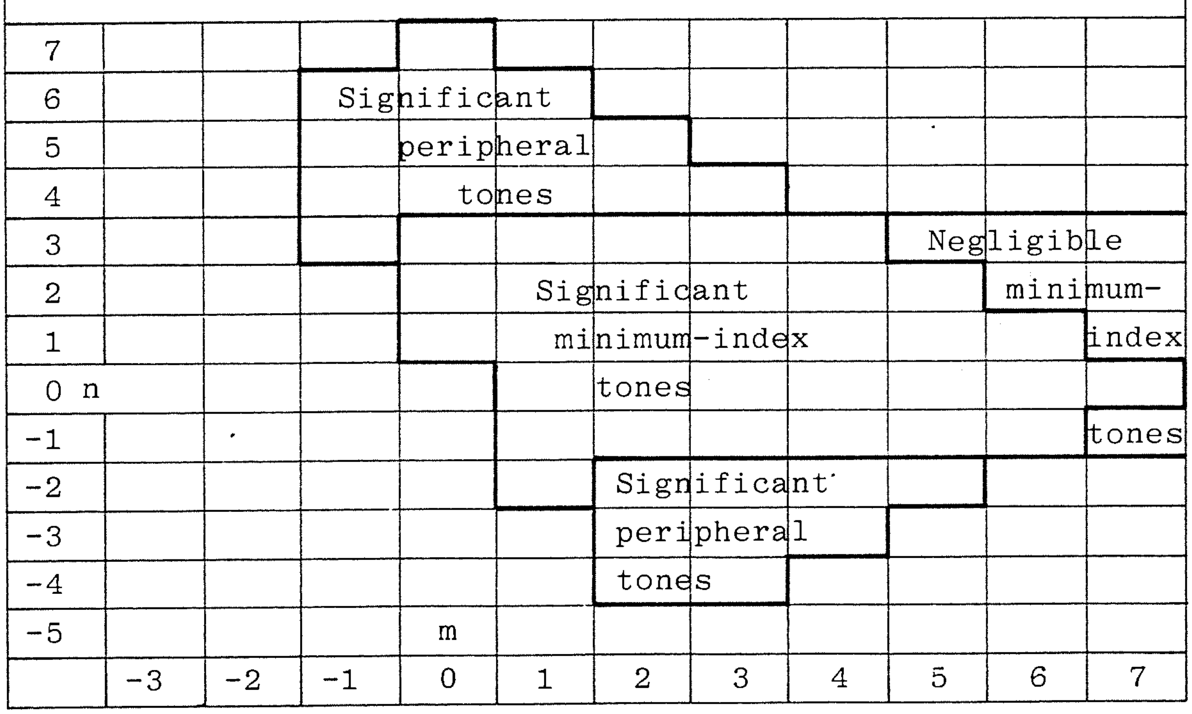
The continuity of the significant tones is broken when a frequency less than Nx is supplied only by negligible tones, that is, tones for which

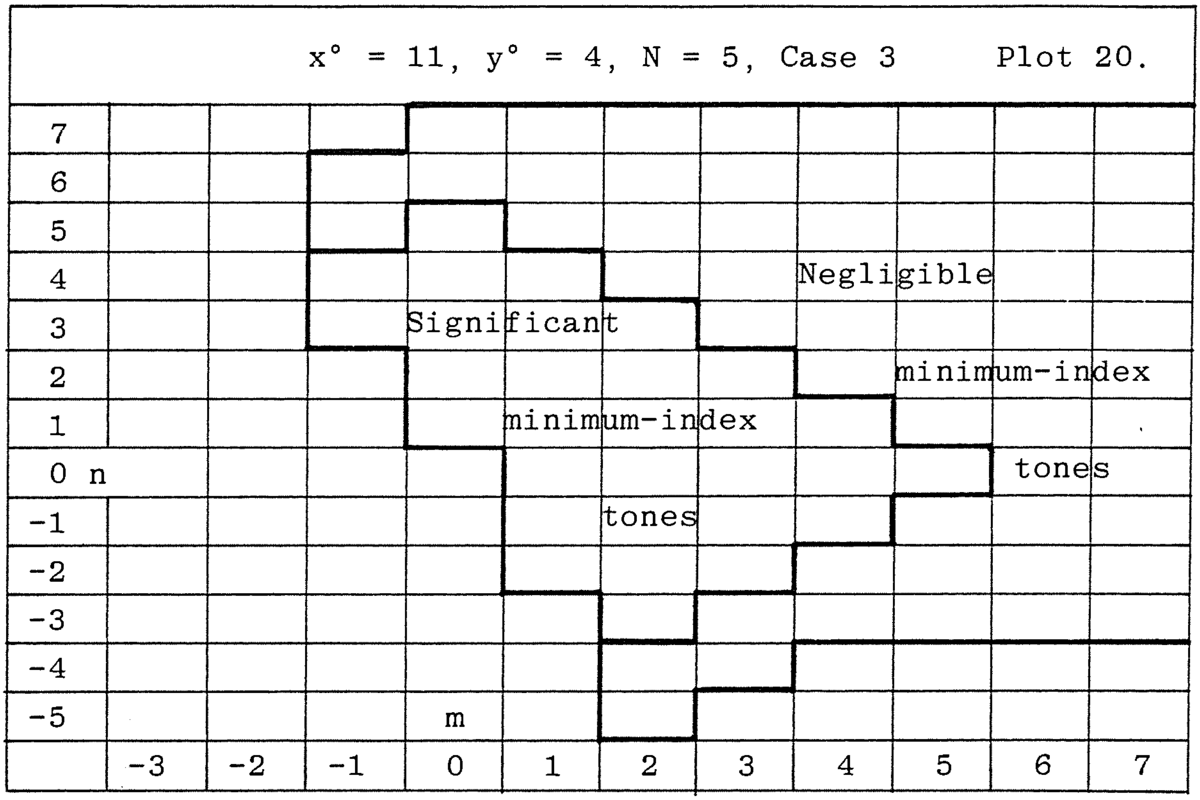
$$G > N \text{ or } 2G > 2N + 1 \qquad \text{F4}$$

Plot 18. $x^\circ = 8$, $y^\circ = 3$, $N = 5$, Case 1



Plot 19. $x^\circ = 5$, $y^\circ = 2$, $N = 7$, Case 2





G being understood to designate the index whether it is G or G'. There is a minimum-index tone for every frequency of the spectrum; therefore, the continuity of the significant tones can be broken only when a minimum-index tone is negligible. Furthermore, any negligible minimum-index tone with $f < Nx$ breaks the continuity of the significant tones, because any other tone of the same frequency has an equal or larger index and is also negligible. It follows that the significant tones are continuous up to but not including the lowest negligible minimum-index tone.

<u>Cases</u> <u>1</u> <u>and</u> <u>2</u>. F1, F2, and F4 unite to give

$$2G > 2N + 1 \geqq x^\circ + y^\circ \qquad \text{F5}$$

but F4 is the condition for tones to be negligible; therefore, in these two cases, all negligible tones have indexes greater than $(x^\circ + y^\circ)/2$. This includes negligible minimum-index tones; but minimum-index tones with $G > (x^\circ + y^\circ)/2$ are in either the first or fourth quadrant; therefore, in Cases 1 and 2, any negligible minimum-index tone is in the first or fourth quadrant.

When primes are applied to f, G, and n, B4 gives us the following expression for the frequency of the tone in the first quadrant:

$$f' = G'x - n'(x - y) \qquad \text{F6}$$

The lowest tone in the first quadrant with a given minimum index results when the largest value of n' is taken from E11, namely

$$n' = (x^\circ + y^\circ - \mu)/2 \qquad \text{F7}$$

where

$$\begin{aligned} \mu &= 0 \text{ when } x^\circ \pm y^\circ \text{ is even} \\ &= 1 \text{ when } x^\circ \pm y^\circ \text{ is odd.} \end{aligned} \qquad \text{F8}$$

B13 gives us the following expression for the frequency of the tone in the fourth quadrant:

$$f = Gx + n(x + y) \qquad \text{F9}$$

The lowest tone in the fourth quadrant with a given minimum index results when the smallest value of n is taken from E8, namely

$$n = -(x^\circ - y^\circ - \mu)/2 \qquad \text{F10}$$

The indexes of these two tones must be equal, but their frequencies may be different.

Letting G' = G, we subtract F6 from F9 and, with the aid of F7, F10, and A5, obtain

$$\begin{aligned} f - f' &= (n' + n)x - (n' - n)y \\ &= y^\circ x - (x^\circ - \mu)y \\ &= \mu y \end{aligned} \qquad \text{F11}$$

It follows that f' is the frequency of the lowest minimum-index tone with a given index greater than $(x^\circ + y^\circ)/2$. Both tones have the same frequency when $x° \pm y°$ is even.

The frequency of the lowest negligible minimum-index tone is, therefore, given by F6 when n' is given by F7 and G' is given the smallest value possible under the conditions. Since F4 is the governing condition, G' = N + 1, and this frequency is

$$f' = (N + 1)x - n'(x - y) \qquad \text{F12}$$

$$= x + Ny + (N - n')(x - y) \qquad \text{F13}$$

Solving F1 for N and comparing the result with F7 yields

$$N = (x° + y° - \mu)/2 = n' \qquad \text{F14}$$

therefore, in Case 1 or when x = y,

$$f' = x + Ny \qquad \text{F15}$$

F2 and F7 give

$$N > (x° + y°)/2 \geqq n' \qquad \text{F16}$$

therefore, in Case 2 when x > y,

$$f' > x + Ny \qquad \text{F17}$$

In both cases it is always true that

$$f' \geqq x + Ny \qquad \text{F18}$$

Since f' is the frequency of the first gap, this shows that the range of continuity extends up to and possibly beyond $x + Ny$ when $x° + y° \leqq 2N + 1$.

F12 can also be written in the familiar form $f' = m'x + n'y$ where

$$m' = N - n' + 1 \geqq 1 \qquad \text{F19}$$

The application of this to certain intervals is shown below:

$x° + y°$	2	3	4	5	6	7
$x°/y°$	1/1	2/1	3/1	3/2, 4/1	5/1	4/3, 5/2, 6/1
f'	Nx + y		(N - 1)x + 2y		(N - 2)x + 3y	

F20

$x^\circ + y^\circ$	8	9	10	11
x° / y°	5/3, 7/1	5/4, 7/2, 8/1	7/3, 9/1	6/5, 7/4, 8/3
f'	(N - 3)x + 4y		(N - 4)x + 5y	

In accord with this, the first gap (that is, the lowest break in the continuity of the significant tones) in Plots 16 and 17 is at 2x + 5y.

The frequency of the highest significant tone is Nx; and the continuity of the significant tones is broken if, and only if, $f' < Nx$. Therefore, it is important to compare f' with Nx. This is done by calculating $f' - Nx$. Subtracting Nx from f' as given in F12 results in

$$\begin{aligned} f' - Nx &= x - n'(x - y) \\ &= x - (x^\circ + y^\circ - \mu)(x - y)/2 \\ &= x - (2y^\circ + x^\circ - y^\circ - \mu)(x - y)/2 \\ &= x - (x^\circ - y^\circ)y - (x^\circ - y^\circ - \mu)(x - y)/2 \end{aligned} \qquad \text{F21}$$

which can be used to calculate the correct value of f' but not the correct values of m' and n'. It is interesting to note the dependence of $f' - Nx$ on $x^\circ - y^\circ$. A tabulation of expressions for $f' - Nx$ for several values of $x^\circ - y^\circ$ follows:

$x^\circ - y^\circ$	0	1	2	3	4	5	6
$f' - Nx$	x	x - y	-y	-2y	-x -2y	-x -3y	-2x -3y

F22

When $x^\circ - y^\circ = 0$, $x^\circ = y^\circ = 1$, and $f'^\circ - Nx^\circ = x^\circ = 1$. When $x^\circ - y^\circ = 1$, $x^\circ/y^\circ = (y^\circ + 1)/y^\circ$, a superparticular ratio, and $f'^\circ - Nx^\circ = x^\circ - y^\circ = 1$. This shows us that, in the unison and the intervals with superparticular frequency ratios (such as the octave, the perfect fifth, the perfect fourth, and the major and minor thirds in just intonation), the first gap is the first tone above Nx or, in other words, the significant tones are continuous thru Nx. In other intervals, gaps occur below Nx. These observations are found to be in exact agreement with Figure 4, Chapter 2, which shows the significant tones corresponding with $N = 6$.

Case 3. There are two ways to satisfy both F3 and F4.

First, $2G > x° + y° > 2N + 1$ F23

Second, $x° + y° \geqq 2G > 2N + 1$ F24

In the first way, as in Cases 1 and 2, the negligible tones have indexes greater than $(x° + y°)/2$. As a result, the negligible minimum-index tones are in the first and fourth quadrants, F6 thru F11 still apply, and the frequency of the lowest negligible minimum-index tone is given by F6 when n' is given by F7 and G' is given the smallest permitted value. Unlike Cases 1 and 2, however, the value of G' (which equals G) is restricted by F23, which also takes the form

$$G' > n' > N \qquad \text{F25}$$

In consequence, the smallest permitted value of G' is $n' + 1$; and, by substitution in to F6, the first way gives

$$\begin{aligned} f' &= (n' + 1)x - n'(x - y) \\ &= x + n'y \\ &> x + Ny \qquad \text{F26} \end{aligned}$$

The second way to satisfy F3 and F4 always offers at least one index less than or equal to $(x° + y°)/2$ and greater than N, and only negligible minimum-index tones can have such an index. It was noted at the end of Section B that the lowest tones for a given index are in the second and fourth quadrants and fall within the basic range; that is, $f \leqq x + y$ and $f' \leqq x + y$. A comparison of this with F26 shows clearly that the second way offers lower negligible minimum-index tones than the first way.

Thus the lowest negligible minimum-index tones of Case 3 are those found in the second way, and they always introduce at least one gap in the basic range of the aural spectrum. This spoils the continuity of the significant tones even within a more restricted range than g to x + Ny. Since, in this case, $x^\circ + y^\circ > 2N + 1$, it can be seen that F5 is a necessary as well as a sufficient condition for the first gap (or lowest break) in the significant portion of the spectrum to fall at or above x + Ny.

G. Primary Coincidences

Of primary importance are the coincidences obtained when $G' + G$ and $|G' - G|$ are minimal under the conditions imposed by D14. This gives

$$k = 1 \qquad \text{G1}$$

$$G' + G = x^\circ + y^\circ \qquad \text{G2}$$

$$|G' - G| = \mu \qquad \text{G3}$$

where μ is as defined in F8.

G1 identifies the coinciding tones as being adjacent; G2 identifies the case as 24, thereby placing $m'x + n'y$ in the second quadrant and $mx + ny$ in the fourth; and G3 shows that the two tones are of equal or nearly equal loudness. If $x^\circ + y^\circ$ is even, $\mu = 0$ and both indexes are a minimum. If $x^\circ + y^\circ$ is odd, $\mu = 1$, one of the indexes is a minimum, and the other is the next smallest for tones of their frequency. Therefore, the two tones are the two strongest of their frequency. The coincidences thus provided must be the best obtainable, that is, those that are of the greatest aural significance for the interval. For this reason, they will be referred to as primary coincidences. Coincidences of other tones with these or with each other will be called secondary coincidences.

Inasmuch as they belong to Case 24, primary coincidences involve only partials and difference tones, which are generally lower in frequency than summation tones. This agrees with identifying them as the most significant coincidences, for experimental evidence has found that summation tones appear to be weaker than difference tones of the same degree and that the lower tones may mask the higher tones (Ch. 2, Sec. F and G).

G3 can also be expressed as

$$G' - G = \pm \mu \qquad \text{G4}$$

Adding this to G2 and dividing the resulting sum by 2 gives

$$G' = (x° + y° \pm \mu)/2 \qquad \text{G5}$$

The procedure set forth in Section D for determining the coefficients of the coinciding tones requires in Case 24 that we compare G with ky° or G' with kx°. We choose the latter and, since k = 1 here, compare G' with x°. Because of A6, $2x° \geqq x° + y° \geqq 2y°$ or $2y° \leqq x° + y° \pm \mu \leqq 2x°$, and this together with G5 yields

$$y° \leqq G' \leqq x° \qquad \text{G6}$$

In view of this, D21 or D22 may be used to choose values of m' or n'. Let D21 be used to find values of m'; then, for each m', D2, C9, and C10 provide the following expressions for the remaining coefficients:

$$n' = G' + m' \qquad \text{G7}$$
$$m = m' + y° \qquad \text{G8}$$
$$n = n' - x° \qquad \text{G9}$$

Substitution from G7 into C1 results in

$$\begin{aligned} f' &= m'x + (G' + m')y \\ &= m'(x + y) + G'y \end{aligned} \qquad \text{G10}$$

which is equivalent to B7 with primes applied to f, m, and G. G5 gives one value of G' when x°+ y° is even and two values when x°+ y° is odd. For each value of G', the lowest value of m' obtainable under D21 is also the lowest value

Plot 21. $x^\circ = 3$, $y^\circ = 2$, Primary coincidences.

7											
6											
5											
4											
3				6							
2			1	4							
1											
0 n						6					
-1					1	4					
-2											
-3											
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

that results in a positive frequency. This places the coinciding tones in the basic range, and identifies them as minimum-frequency tones. Inasmuch as there are one or two values of G', there are one or two corresponding primary coincidences in the basic range. The primary coincidences for three intervals together with boundaries locating the minimum-index tones are shown in Plots 21 thru 23. Those for two of these plus four other intervals that can reasonably well be given by tones of our musical scale are shown in Figure 1.

As noted in Chapter 1, Section D, the lowest coincidence of two partials occurs between $x^\circ y$ and $y^\circ x$. Letting $m' = 0$, $n' = x^\circ$, $m = y^\circ$, and $n = 0$, we see that this is a Case 24 coincidence with $G' = x^\circ$ and $G = y^\circ$. In order for this to be a primary coincidence, G2 and G3 must be x° = 5, y° = 1, Primary coincidence. Plot 22.

7											
6											
5											
4											
3				3							
2											
1											
0 n											
-1											
-2					3						
-3											
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

x° = 5, y° = 4, Primary coincidences. Plot 23.

7											
6											
5				20							
4			11	16							
3		2	7								
2											
1											
0 n								20			
-1							11	16			
-2						2	7				
-3											
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

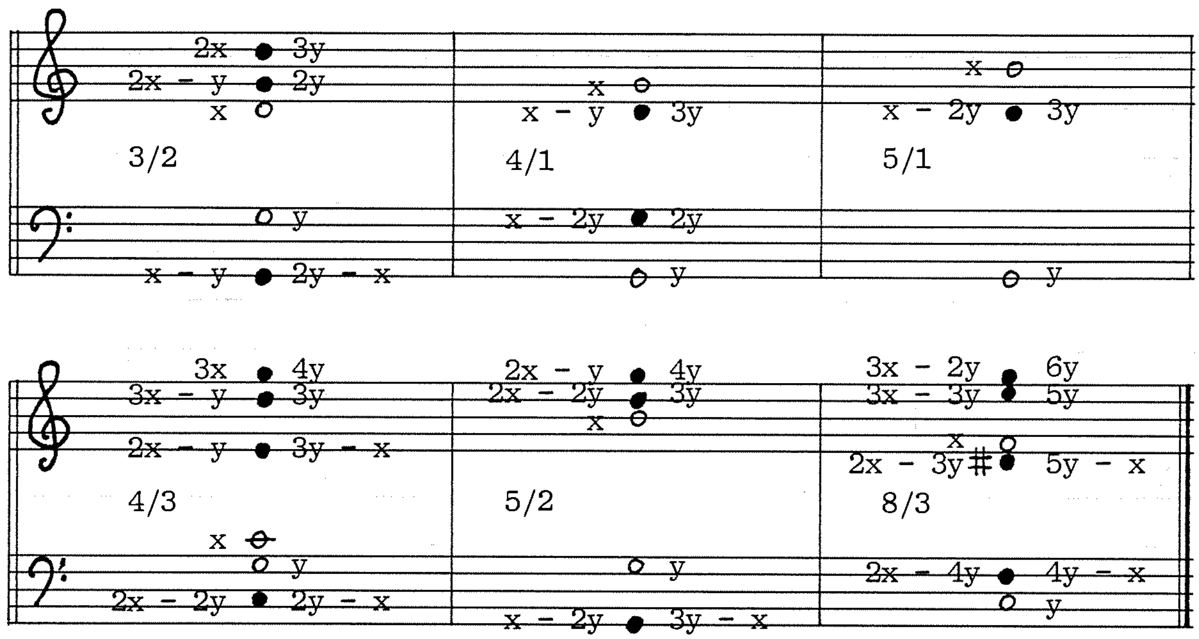


Figure 1. Primary coincidences. Just intonation is required for the coincidences to be exact; even so, the d# given for 8/3 is only approximate. The white notes represent the fundamentals of the generating tones; and the black notes, the coinciding partials and difference tones.

satisfied. G2 is already satisfied, and G3 is satisfied when $x^\circ - y^\circ = \mu$. In other words, the lowest coincidence of two partials is a primary coincidence only for the unison and the superparticular ratios, esteemed by Ptolemy (second century A.D.) and many others since then.[2] Examples of such appear in Plots 21 and 23.

Substitution from G5 into G10 gives

$$f' = m'(x + y) + (x^\circ + y^\circ \pm \mu)y/2 \qquad \text{G11}$$

The highest value of m' obtainable under D21 is 0; therefore, the frequency of the highest primary coincidence is

$$f' = (x^\circ + y^\circ + \mu)y/2 \qquad \text{G12}$$

---

2. J.M. Barbour, Tuning and Temperament, pp. 2 and 23.

In Section F, f' is the frequency of the lowest negligible minimum-index tone, and there F14, F16, and F18 give

$$f' \geqq x + (x^\circ + y^\circ - \mu)y/2$$

Subtraction of G12 from this results in a difference greater than or equal to $x - \mu y$, a positive quantity. When $x^\circ + y^\circ \leqq 2N + 1$, then, all the primary coincidences clearly fall within the range of continuity. In such coincidences, at least one of the tones is significant while the other may be negligible as in Plot 23 with N = 4.

H. Coincidence of Significant Spectral Tones

For two spectral tones to be significant, it is required that

$$G' \leqq N \text{ and } G \leqq N \qquad \text{H1}$$

These two requirements can be condensed into the one convenient statement

$$G' + G + |G' - G| \leqq 2N \qquad \text{H2}$$

Combining this with D14 results in

$$|G' - G| \leqq k(x^\circ + y^\circ) \leqq G' + G \leqq 2N - |G' - G| \qquad \text{H3}$$

which expresses the necessary and sufficient conditions for the coincidence of significant tones of the aural spectrum.

Foremost among these conditions is

$$x^\circ + y^\circ \leqq 2N \qquad \text{H4}$$

As long as this requirement is satisfied, values of k, G, and G' can be found that conform to H3 and consequently determine coincidences in which the tones are significant. Intervals that conform to H4 have aural spectra in which significant tones coincide, and intervals that do not conform to this do not exhibit the coincidence of significant tones. When a specific value is assigned to N, H4 places an upper limit on $x^\circ + y^\circ$; thus, the value of N regulates <u>which</u> ratios of x to y exhibit the coincidence of significant tones of the aural spectrum. All the intervals distinguished by the coincidence of significant tones when N = 6 can be found in Figure 4, Chapter 2. These intervals demonstrate adherence to H4 in that none of them has a ratio of x to y in which $x° + y°$ exceeds 12.

When $x° + y°$ is as close as possible to the upper limit permitted by H4, $x° + y° = 2N - \mu$

Then, by H3, $k = 1$, $G' + G = x° + y°$, $|G' - G| = \mu$, and there are only primary coincidences. Conversely, substitution from G1, G2, and G3 into H3 gives

$$\mu \leqq x° + y° = G' + G \leqq 2N - \mu$$

which shows that the primary coincidences satisfy the conditions for the coinciding tones to be significant when H4 is respected.

When $x° + y° < 2N - \mu$, there are both primary and secondary coincidences of significant tones. As an example, let us consider the just major third with N = 6. Here $x° = 5$, $y° = 4$, $x° + y° = 9$, $2N = 12$, and k can only equal 1. Substitution of these quantities into H3 yields

$$|G' - G| \leqq 9 \leqq G' + G \leqq 12 - |G' - G|$$

This and the requirement that $G' + G$ and $|G' - G|$ must differ from $k(x° + y°)$ by even numbers give the results tabulated below:

G'+ G	G'- G	G'	G	Coincidences
9	1	5	4	Primary
9	-1	4	5	
9	3	6	3	Secondary
9	-3	3	6	
11	1	6	5	
11	-1	5	6	

x° = 5, y° = 4, N = 6, Secondary coincidences. Plot 24.											
7											
6				24							
5			15		25						
4		6			21						
3				12							
2			3								
1								24			
0 n							15		25		
-1						6			21		
-2								12			
-3							3				
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

These values of G' and G used in conjunction with the procedures of Section D lead to the five primary coincidences of Plot 23 and the seven secondary coincidences of Plot 24. All twelve coincidences of significant tones can be seen in Figure 4, Chapter 2.

I. The Basic Range

The prominence of the parameter $x° + y°$ prompts us to observe that it is the number of distinct frequencies in the basic range (g to x + y inclusive). This portion of the aural spectrum embraces the lowest tones, including x and y, and the one or two lowest primary coincidences. Furthermore, the continuity of this range is the key to the continuity of the spectrum. If the spectrum is continuous in the basic range, it is continuous thru a larger range; if the spectrum is not continuous in this range, it is not continuous. As we shall see, the number of significant tones in this range is 2N + 1 regardless of the ratio of x to y, whether commensurable or incommensurable.

The tone x + y is the lowest summation tone, and there is no other summation tone as low as this one. Therefore, any other spectral tone as low as or lower than x + y must be a partial or a difference tone; that is, any minimum-frequency tone other than x + y must be in the second or fourth quadrant. As was observed in Section B, there are one minimum-frequency tone in the second quadrant and one in the fourth quadrant for every value of G. Since we are considering only significant tones, G consists of the N integers in the range $1 \leqq G \leqq N$. It follows that there is a total of 2N significant minimum-frequency partials and difference tones. Inclusion of the summation tone x + y requires that $N \geqq 2$; and, with this proviso, the number of significant spectral tones in the basic range equals 2N + 1.

Let us consider how these 2N + 1 significant tones are distributed among the $x^\circ + y^\circ$ distinct frequencies of the basic range. When $x^\circ + y^\circ = 2N + 1$, there is no coincidence of significant tones, the frequency of the first gap is x + Ny, and it follows that there is exactly one significant tone for every distinct frequency in the range below x + Ny. If coincidences exist, there are no gaps; if gaps exist, there are no coincidences. In other words, the distribution is as nearly even as possible.

Further observations confirm that this principle of even distribution can be generalized and extended as follows. Division of 2N + 1 by $x^\circ + y^\circ$ results in a quotient q and a remainder r equal to $2N + 1 - q(x^\circ + y^\circ)$, which is less than $x^\circ + y^\circ$. Then there are q significant tones for each of

$x° = 4$, $y° = 3$, $N = 7$
Significant minimum-frequency tones. Plot 25.

7											
6											
5		7									
4		4									
3		1	5								
2			2	6							
1				3	7						
0 n					4						
-1					1	5					
-2						2	6				
-3							3	7			
-4											
-5				m							
	-3	-2	-1	0	1	2	3	4	5	6	7

$x° + y° - r$ distinct frequencies and $q + 1$ significant tones for each of r distinct frequencies within the basic range. In particular, there are q significant tones for each frequency when $r = 0$, and there are r coincidences of significant tones when $q = 1$. These relations can be observed in Plots 16 and 17 and are specifically illustrated in Plot 25. In Plot 16, $q = 1$, $r = 2$, and there are two coincidences of significant tones in the basic range. In Plot 17, $q = 1$, $r = 3$, and there are three coincidences of significant tones in the basic range. In Plot 25, $q = 2$, $r = 1$, two significant tones coincide at each of six minimum frequencies, and three tones coincide at the frequency of $x + y$.

J. Discernible Intervals

A summary of the relations presented so far concerning continuity and coincidence of the audibly significant tones of the aural spectrum is tabulated below:

First Gap	$x^\circ + y^\circ$	Coincidences
Above $x + Ny$	$< 2N - 1$	Primary and secondary
$N > (x^\circ + y^\circ - \mu)/2$	$= 2N - 1$	Primary only
At $x + Ny$	$= 2N$	$N = (x^\circ + y^\circ + \mu)/2$
$N = (x^\circ + y^\circ - \mu)/2$	$= 2N + 1$	None
Below $x + y$	$> 2N + 1$	$N < (x^\circ + y^\circ + \mu)/2$

The following paragraph offers assistance in the interpretation of this tabulation.

When $x^\circ + y^\circ = 2N + 1$, the significant tones have neither break nor coincidence within the range starting with g and extending up to but not including $x + Ny$. Thus, in Figure 4, Chapter 2, in which N is 6, all the intervals for which $x^\circ + y^\circ$ equals 13 have exactly one significant tone to every distinct frequency within the range from g to $x + 6y$. When $x^\circ + y^\circ = 2N$, the range of continuity is still the same, altho primary coincidences will now be found within this range. When $x^\circ + y^\circ = 2N - 1$, F7 gives $n' = N - 1$, F13 places the first gap at $2x + (N - 1)y$, and there are still only primary coincidences within the range of continuity.

Two general conclusions follow:

1. Intervals in which $x^\circ + y^\circ$ exceeds $2N + 1$ do not have aural spectra that exhibit audibly significant continuity.
2. There is no audibly significant coincidence of tones in the spectra of intervals in which $x^\circ + y^\circ$ exceeds $2N$.

Now continuity and coincidence of spectral tones were shown at the beginning of this chapter to be characteristic of

Table I

The Discernibly Commensurable Intervals

The intervals are listed in the order of decreasing discernibleness of continuity and coincidence of spectral tones. Names that apply strictly only to intervals one or more octaves smaller are in parentheses.

$x^\circ + y^\circ$	x° / y°	Name
2	1/1	unison
3	2/1	octave
4	3/1	per. 12th
5	3/2	per. 5th
	4/1	15th
6	5/1	(maj. 3rd)
7	4/3	per. 4th
	5/2	maj. 10th
	6/1	(per. 5th)
8	5/3	maj. 6th
	7/1	
9	5/4	maj. 3rd
	7/2	
	8/1	(octave)
10	7/3	
	9/1	(maj. 2nd)
11	6/5	min. 3rd
	7/4	
	8/3	per. 11th
	9/2	(maj. 2nd)
	10/1	(maj. 3rd)
12	7/5	
	11/1	
13	7/6	
	8/5	min. 6th
	9/4	maj. 9th
	10/3	maj. 13th
	11/2	
	12/1	(per. 5th)
14	9/5	min. 7th
	11/3	
	13/1	
15	8/7	
	11/4	
	13/2	
	14/1	
16	9/7	
	11/5	
	13/3	
	15/1	(maj. 7th)

commensurable intervals, but we have found that they are audibly discernible only when $x^\circ + y^\circ$ is limited in magnitude to 2N or less. Let us say that an interval is discernibly commensurable (or simply discernible) when $x^\circ + y^\circ \leqq 2N$. Moreover, these discernible intervals do not all demonstrate the continuity and coincidence of significant spectral tones to an equally discernible extent. Smaller values of $x^\circ + y^\circ$ result in the continuity and coincidence of spectral tones that have smaller indexes and are therefore stronger. For this reason, the discernible intervals will be said to be *more* discernible when $x^\circ + y^\circ$ is smaller, and *less* discernible (or more obscure) when $x° + y°$ is larger.

All the intervals that are discernible with N = 8 are listed in Table I, where they are classified in the order of the values of $x° + y°$ resulting from their ratios. Intervals that fall in the same class according to this measure are further differentiated according to the value of $x° - y°$. Thus, the intervals can be said to be listed in the order of decreasing discernibleness (or of increasing obscurity), those at the beginning being easily discernible and those at the end being discernible only with difficulty.

K. The "Third Tone"

In Chapter 2, Section E, it was pointed out that the aural spectrum of a musical tone consists of pure tones (or partials) whose frequencies fall according to the ratios of the harmonic series (that is, x, 2x, 3x, 4x, and so forth) and that the frequency of the first partial (or fundamental) is aurally interpreted as being that of the tone as a whole. Now, the aural spectrum of a discernible interval also consists of pure tones whose frequencies fall according to the ratios of the harmonic series (that is, g, 2g, 3g, 4g, and so forth); and the frequency of the lowest of these tones, which may also be called the fundamental, is aurally significant as the identifying frequency of the spectrum as a whole. Thus, the aural spectrum of a discernible interval is identical in form to that of a single musical tone, and we can truly say that it constitutes a "third tone" that is heard with every such interval, whose discernibleness is proportional to that of the interval itself, and which is related in frequency to the two primary tones as fundamental to partials.

In spite of the fact that the third tone is well represented in that it usually has more "partials" than the typical musical tone, it is not always independently perceived, and there has been confusion over the pitch relationship of the third tone to the primary tones, which overshadow it in the hearing and thus cloud the perception and identity of the third tone. Nevertheless, each primary tone is heard as having a certain harmonic quality peculiar to its position in the spectrum, that is, according to which partial of the third tone it is. Thus, any primary tone of an interval that is related to the other tone as a fundamental to an upper partial is coincident with the fundamental of the third tone and is heard as having the harmonic quality of a fundamental; any primary tone of an interval that is related to the other primary tone as a second partial to another partial is coincident with the second partial of the third tone and is heard as having the harmonic quality of a second partial; and so forth. That the primary tones of discernible intervals have harmonic qualities determined by the ratio of x to y agrees with the well known fact that every discernible interval has a distinguishing (or characteristic) sound that is detected by the musician's ear and plays a major role in the recognition of specific harmonic intervals.

Apparently Tartini was the first to make the discovery of the third tone altho not the first to publish it.[3] Shirlaw paraphrases Tartini thus:

> If, he points out, two sounds of just intonation be sounded clearly and loudly together, there will result a third sound, lower in pitch than the other two, and which will be the fundamental[1] sound of the harmonic series of which the first two sounds form an integral part.

---

3. Giuseppe Tartini's published account was in his Trattato di musica secondo la vera scienza dell' armonia, 1754. Two earlier publications describing such tones were by Sorge in 1744 and by Romieu in 1751.

and adds this footnote:

> 1. Tartini, however, does not here say fundamental, but octave of the fundamental, corresponding to the term ½, and in the examples he gives of the resultant "third sound," places it an octave too high. This mistake he afterwards corrected.[4]

The historically recent discovery of the third tone is cause for comment, but its existence can hardly be doubted when one considers the continuity of an aural spectrum that has coincidences and recognizes that the tunableness of an interval by the elimination of beats (Ch. 2, Sec. B) demonstrates the coincidence of spectral tones. That most of the intervals of Table I are thus tunable appears to be substantiated by experimental results of Harry Partch. He says:

> Experience in tuning the Chromelodeon has proved conclusively that not only the ratios of 3 and 5, but also the intervals of 7, 9, and 11 are tunable by eliminating beats.[5]

He says also:

> . . . the ear (or this ear) cannot possibly hear 24 to 25 as such, even though the two tones definitely preserve their individual integrity (the question Meyer raises), for if one is slightly out of tune, so that the ratio is, let us say, 24.1 to 25, the beats will simply be a trifle slower. Yet in a median register on the Chromelodeon both 9/8 and 10/9 are clearly heard, by tuning them to eliminate beats and by the establishment of high frequency of wave period in 8 to 9 and 9 to 10. Almost anyone can distinguish and tune 6/5, which is the only 5-limit ratio in this particular range of ratios. Consequently, at some point, to be determined by the individual, in the narrowing of intervals

---

4. Matthew Shirlaw, The Theory of Harmony, p. 289.

5. Genesis of a Music, p. 139. By "the intervals of 7, 9, and 11" he means intervals whose frequency ratios can be expressed by integers respectively not larger than 7, 9, and 11, except that either term of the ratio may be multiplied by any reasonably small positive integral power of 2. The Chromelodeon is a modified reed organ whose tone "has a very rich harmonic content."

> between the wide 6/5 and the narrow 16/15 or 25/24, and at the corresponding point in the approach to 2/1 (and, incidentally, at some prime number larger than 11 in the wider intervals), the ear refuses to distinguish between rational and irrational numbers.[6]

Partch's "intervals of 7, 9, and 11" are amply in evidence in Table I in such ratios as 7/3, 8/7, 9/5, 11/4, and so forth. The interval 9/8, which he intimates is among the more difficult intervals to tune by ear, does not appear in Table I and therefore would be expected to be difficult, if not impossible, to tune by the elimination of beats. It is especially remarkable that we find with Partch that prime numbers larger than 11 occur in intervals larger than an octave.

The author, who has had experience in tuning pianos, finds that he is able to tune all the intervals that he can play with one hand and for which $x° + y° \leqq 16$ by the elimination of beats. This includes most of Partch's intervals except 9/8 and 10/9, which would naturally be easier to tune on a Chromelodeon than on a piano. The difficulty of tuning the intervals increases as $x° + y°$ increases and as one leaves the middle range of the piano, but the tunableness of these intervals by the elimination of beats stands as evidence of the existence of a definite third tone not only in the usual intervals of just intonation (3/2, 4/3, 5/3, 5/4, 6/5) but also in many others that are not now recognized in musical theory or practice.

The historically early recognition in Western music of the easily discernible intervals 1/1, 2/1, 3/2, and 4/3; the later recognition of 5/3, 5/4, 6/5, and 8/5; and the recognition not yet generally achieved of 7/4, 7/5, 7/6, and less discernible intervals strongly suggest that discernibleness

---

6. Ibid., pp. 148-149.

is an important deciding factor in the selection of intervals for musical purposes. It is the author's belief that the discernibly commensurable intervals appeal either consciously or intuitively to musicians in their search for usable harmonic intervals and that, therefore, all such intervals are capable of harmonic usage in music. If this is so, the recognition in theory and the adoption in practice of such intervals as 7/3, 7/4, and 7/5 is at present a logical and desirable avenue of advance in the development of greater harmonic resources. The principal difficulty is finding the technical means to handle them. It is perhaps equally significant, however, that the number of discernible intervals is limited and that, therefore, the development of music in the direction of new basic harmonic relationships would eventually meet a stopping point in whatever forward movement it might enjoy in the future.

Chapter 4

THE INTERPRETATION OF INTERVALS

A. Interrelating Commensurable Intervals

Using the smallest positive integers that can express the ratios of the frequencies of the intervals, let x'/y' be the frequency ratio of a given commensurable interval, let x''/y'' be that of a given larger interval, let $x°/y°$ be that of any other commensurable interval, and let

$$i = y'x'' - x'y'' \qquad A1$$

$$i' = y°x'' - x°y'' \qquad A2$$

$$i'' = y'x° - x'y° \qquad A3$$

These determinants have integral values because they are composed of integers; and, since division of A1 by $y'y''$ gives $i/(y'y'') = x''/y'' - x'/y'$, it is seen that i is positive because $x'/y' < x''/y''$.

When $x°/y° < x'/y'$, $i' > 0$ and $i'' < 0$. A4

When $x°/y° > x''/y''$, $i' < 0$ and $i'' > 0$. A5

When $x'/y' < x°/y° < x''/y''$, $i' > 0$ and $i'' > 0$. A6

If $x°/y° = x'/y'$, $i' = i$ and $i'' = 0$. A7

If $x°/y° = x''/y''$, $i' = 0$ and $i'' = i$. A8

The converses of these are also true. In particular, when i' and i'' are positive, A4 and A5 are impossible and $x°/y°$ is intermediate to x'/y' and x''/y''.

These and many other relationships can be made visibly evident by displaying the various intervals on a single plot in which i' and i'' are the coordinates of $x°/y°$. In such a plot, x'/y', with coordinates $i' = i$ and $i'' = 0$, and x''/y'', with coordinates $i' = 0$ and $i'' = i$, serve as base intervals to establish the locations of the other intervals. This is exemplified in Plot 1, where it is easily seen that intervals smaller than x'/y' are in the fourth quadrant, intervals larger than x''/y'' are in the second quadrant, and intervals of intermediate size are in the first quadrant.

Division of A3 by A2 results in

$$\frac{i''}{i'} = \frac{y'x^\circ - x'y^\circ}{y^\circ x'' - x^\circ y''} = \frac{(x^\circ/y^\circ - x'/y')y'}{(x''/y'' - x^\circ/y^\circ)y''} \qquad \text{A9}$$

This shows that i''/i' is a function of x°/y° such as to increase from zero to infinity as x°/y° increases from x'/y' to x''/y''. It is also evident that i''/i' is negative when x°/y° is less than x'/y' or greater than x''/y''. A vector (directed line segment) from the origin (where $i' = i'' = 0$) to an interval makes an angle with the horizontal (or i') axis whose tangent is i''/i'. This associates a certain direction with each interval as seen from the origin. We note that no two intervals have the same direction, for, if they did, they would have the same frequency ratio.

From A1, A2, and A3, we obtain

$$x'i' + x''i'' = x^\circ i \qquad \text{A10}$$

$$y'i' + y''i'' = y^\circ i \qquad \text{A11}$$

Inasmuch as x° and y° are relatively prime, these equations show us that i is the greatest common divisor of $x'i' + x''i''$ and $y'i' + y''i''$; but a common factor of i' and i'' is a common divisor of $x'i' + x''i''$ and $y'i' + y''i''$ and therefore a factor of i. Transforming A10 and A11 to

$$x''i'' = x^\circ i - x'i' \qquad \text{A12}$$

$$y''i'' = y^\circ i - y'i' \qquad \text{A13}$$

shows us that i'' is the greatest common divisor of $x^\circ i - x'i'$ and $y^\circ i - y'i'$ and that a common divisor of i and i' is a factor of of i''. Similarly, a common divisor of i

$\frac{x'}{y'} = \frac{2}{1}, \frac{x''}{y''} = \frac{3}{1}, i = 1$											Plot 1.
6				$\frac{16}{5}$		$\frac{20}{7}$				$\frac{28}{11}$	
5	$\frac{7}{1}$	$\frac{9}{2}$	$\frac{11}{3}$	$\frac{13}{4}$		$\frac{17}{6}$	$\frac{19}{7}$	$\frac{21}{8}$	$\frac{23}{9}$		$\frac{27}{11}$
4		$\frac{6}{1}$		$\frac{10}{3}$		$\frac{14}{5}$		$\frac{18}{7}$		$\frac{22}{9}$	
3			$\frac{5}{1}$	$\frac{7}{2}$		$\frac{11}{4}$	$\frac{13}{5}$		$\frac{17}{7}$	$\frac{19}{8}$	
2				$\frac{4}{1}$		$\frac{8}{3}$		$\frac{12}{5}$		$\frac{16}{7}$	
1					$\frac{3}{1}$	$\frac{5}{2}$	$\frac{7}{3}$	$\frac{9}{4}$	$\frac{11}{5}$	$\frac{13}{6}$	$\frac{15}{7}$
i'' 0					0	$\frac{2}{1}$					
-1							$\frac{1}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{4}$	$\frac{9}{5}$
-2										$\frac{4}{3}$	
-3											
-4											
	-4	-3	-2	-1	0 i'	1	2	3	4	5	6

and i" is a factor of i'. Thus, a common factor of any two of these determinants is also a factor of the third; and, if two are relatively prime, all three are relatively prime, in pairs as well as all together. This constitutes what we choose to call a "rule of common factors." A corollary of

Plot 2. $\frac{x'}{y'} = \frac{5}{4}$, $\frac{x''}{y''} = \frac{3}{2}$, i = 2

7	$\frac{3}{1}$		$\frac{8}{5}$		$\frac{13}{9}$		$\frac{18}{13}$		$\frac{23}{17}$		
6								$\frac{19}{14}$			
5			$\frac{5}{3}$		$\frac{10}{7}$		$\frac{15}{11}$				$\frac{25}{19}$
4						$\frac{11}{8}$				$\frac{21}{16}$	
3			$\frac{2}{1}$		$\frac{7}{5}$				$\frac{17}{13}$		$\frac{22}{17}$
2				$\frac{3}{2}$				$\frac{13}{10}$			
1					$\frac{4}{3}$		$\frac{9}{7}$		$\frac{14}{11}$		$\frac{19}{15}$
i" 0				0		$\frac{5}{4}$					
-1					$\frac{1}{1}$		$\frac{6}{5}$		$\frac{11}{9}$		$\frac{16}{13}$
-2								$\frac{7}{6}$			
-3									$\frac{8}{7}$		$\frac{13}{11}$
	-3	-2	-1	0 i'	1	2	3	4	5	6	7

this rule is that i or a factor of i is the only possible common factor of the coordinates of any interval. This is exemplified in Plots 1 and 2, where i is a prime number and is the only common factor of any pair of coordinates.

It can be seen from A1 that a common factor of x' and x" is a factor of i and, consequently, that x' and x" must be relatively prime to each other in order to be relatively prime to i. Again, because of A1, a common factor of i and x' is also a factor of y'x"; but it cannot be a factor of y'; therefore, it must be a factor of x". Likewise, a common factor of i and x" is a factor of x'. Thus the three quantities i, x', and x" obey the rule of common factors. Similarly, i, y', and y" also fall under this rule. Since x' and y' are relatively prime and x" and y" are relatively prime, no common factor of x' and x" can be a factor of y' or y". It follows that a common factor of x' and x" is relatively prime to any common factor of y' and y", but both will be factors of i.

B. Choosing Coordinates when x° and y° are Not Given

A2 and A3 give us the coordinates of x°/y° when numerical values are assigned to x° and y°. Interchanging the sides of A10 and A11 and dividing them by i results in

$$x^\circ = (x'i' + x''i'')/i \qquad \text{B1}$$

$$y^\circ = (y'i' + y''i'')/i \qquad \text{B2}$$

which are the converse of A2 and A3 and therefore give us x° and y° when numerical values are assigned to the coordinates i' and i". Division of B1 by B2 results in

$$\frac{x^\circ}{y^\circ} = \frac{x'i' + x''i''}{y'i' + y''i''} = \frac{x' + x''i''/i'}{y' + y''i''/i'} \qquad \text{B3}$$

which is the inverse of A9 and therefore increases from x'/y' to x"/y" as i"/i' increases from zero (when i" = 0) to infinity (when i' = 0). It shows that there is a commensurable ratio of x° to y° corresponding to any commensurable ratio of i" to i', but B1 and B2 cannot be satisfied by just any integral values of i' and i" that express a given ratio. An arbitrary choice of coordinates could result in values of

$x°$ and $y°$ that are not relatively prime or, if $i > 1$, in fractional values of $x°$ and $y°$, for which only integral values are postulated.

A pair of coordinates that yields values of $x°$ and $y°$ that are both integral and relatively prime is a proper solution (of B1 and B2). A pair of coordinates that yields integral values of $x°$ and $y°$, whether relatively prime or not, is referred to as an integral solution. Coordinates that result in integral values of $x°$ and $y°$ that are not relatively prime constitute an unreduced solution, and those that yield fractional values of $x°$ and $y°$ are called a fractional solution. Unreduced and fractional solutions are to be rejected. To obtain proper solutions, it is necessary for i, i', and i" to obey the rule of common factors; but, even so, there are conditions and choices that can lead to unreduced or fractional solutions.

The rule of common factors permits two extremes. In one extreme, the coordinates are multiples of i, and only i; that is , they are relatively prime except for the common factor of i. This guarantees integral solutions. The other extreme employs coordinates that are relatively prime to each other and to i. When i' and i" do not have a common factor, $x°$ and $y°$ cannot have one, for A2 and A3 show that any common factor of $x°$ and $y°$ would also be common to i' and i". This extreme risks the occurrence of fractional solutions. When $i = 1$, these extremes are united and, to obtain proper solutions, it is only necessary for the coordinates to be relatively prime as in Plot 1. When $i = 2$, all solutions belong to one extreme or the other. Both coordinates must be either even with only 2 as a common factor or odd without a common factor. The only risk is the possibility of unreduced solutions when i' and i" are both even. This can be seen in Plot 2, where $i' = 2$ and $i'' = 6$ would

$\frac{x'}{y'} = \frac{4}{3}$, $\frac{x''}{y''} = \frac{5}{3}$, i = 3 Plot 3.

8		$\frac{12}{7}$						$\frac{20}{13}$			
7	$\frac{9}{5}$			$\frac{13}{8}$			$\frac{17}{11}$				
6						$\frac{14}{9}$					
5		$\frac{7}{4}$			$\frac{11}{7}$						$\frac{19}{13}$
4				$\frac{8}{5}$						$\frac{16}{11}$	
3			$\frac{5}{3}$						$\frac{13}{9}$		
2		$\frac{2}{1}$						$\frac{10}{7}$			
1				$\frac{3}{2}$			$\frac{7}{5}$			$\frac{11}{8}$	
i'' 0			0			$\frac{4}{3}$					
-1					$\frac{1}{1}$			$\frac{5}{4}$			$\frac{9}{7}$
-2										$\frac{6}{5}$	
	-2	-1	0 i'	1	2	3	4	5	6	7	8

give x° = 14 and y° = 10. When i > 2, both dangers exist as we see in Plot 3, where i' = i'' = 3 would give the unreduced solution x° = 9, y° = 6 or i' = 1 and i'' = 2 would give the fractional solution x° = 14/3, y° = 3.

A basic solution is an integral solution for which $0 < i' \leqq i$ and $0 < i'' \leqq i$ and in which i' and i'' are permitted to have a common factor that is not a factor of i. While a basic solution may thus violate the rule of common factors, it does not set the rule aside completely. If one coordinate has a factor in common with i, the other coordinate must also have this factor; and, if one coordinate is relatively prime to i, the other coordinate must also be relatively prime to i. Not requiring that i or a factor of i be the only possible common factor of the coordinates permits unreduced solutions that would otherwise be lost. For each value of i', one value of i'' gives a basic solution; or, for each value of i'', one value of i' gives a basic solution. As a result, there are i basic solutions. All the other integral solutions can be obtained from these by increasing or diminishing either or both coordinates of each basic solution by multiples of i. Finally, unreduced solutions must be rejected, leaving only proper solutions.

One of the basic solutions is always $i' = i'' = i$, whence B1 gives $x° = x' + x''$ and B2 gives $y° = y' + y''$. If $x°$ and $y°$ have a common factor, division of i' and i'' by this factor gives another basic solution; and, if this factor equals i, then $i' = i'' = 1$ is not only a basic solution but also a proper solution and all the basic solutions are those in which $i' = i'' = 1, 2, \ldots i$. This can be seen in Plots 1, 2, and 3 and is always the case when $i = 1$ or 2. The situation may be more complicated when $i > 2$. Let us consider the following example:

$x' = 7,\ x'' = 3$

$y' = 6,\ y'' = 2,\ i = 4$

B1: $x° = (7i' + 3i'')/4$

B2: $y° = (6i' + 2i'')/4 = (3i' + i'')/2$

$i' = i'' = 4$ gives $x° = 10$ and $y° = 8$, which have a common factor of 2. Cancellation of this factor gives the proper solution $i' = i'' = 2$ with $x° = 5$, $y° = 4$. The two remaining basic solutions can only be $i' = 1$, $i'' = 3$ with $x° = 4$, $y° = 3$ and $i' = 3$, $i'' = 1$ with $x° = 6$, $y° = 5$, both proper solutions. All the other integral solutions can be obtained from these four by increasing or diminishing these values of i' and i'' by multiples of i. Finally, all unreduced solutions are discarded, only proper solutions remain, and they are shown in Plot 4.

If x' and x'' are relatively prime and therefore relatively prime to i, a proper solution of B1 also satisfies B2. Let the right side of B1 be substituted for $x°$ in either A2 or A3, and let the resulting equation be solved for $y°x''$ or $x'y°$ and simplified by the use of A1. If substitution is made into A2, the simplified result is

$$y°x'' = (y'i' + y''i'')x''/i \qquad \text{B4}$$

If substitution is made into A3, the result is

$$x'y° = x'(y'i' + y''i'')/i \qquad \text{B5}$$

In both of these results, the left side and therefore the right side is an integer; but, as long as i is relatively prime to both x' and x'', it must divide $y'i' + y''i''$. Thus an integral value of $y°$ is provided, and B4 and B5 reduce to B2 as they should. B2 can be solved in exactly the same way if y' and y'' are relatively prime, and will give the same results. If x' and x'' are not relatively prime, their common factor is also a factor of i, cancellation of this factor simplifies B1, and not every solution of B1 satisfies B2. Likewise, if y' and y'' are not relatively prime, not every solution of B2 satisfies B1. However, since the common factor of x' and x'' is relatively prime to that of y' and y'', there are still i basic solutions.

Plot 4. $\frac{x'}{y'} = \frac{7}{6}$, $\frac{x''}{y''} = \frac{3}{2}$, $i = 4$

8							$\frac{13}{10}$				
7				$\frac{7}{5}$				$\frac{14}{11}$			
6											
5		$\frac{2}{1}$				$\frac{9}{7}$				$\frac{16}{13}$	
4			$\frac{3}{2}$								$\frac{17}{14}$
3				$\frac{4}{3}$				$\frac{11}{9}$			
2					$\frac{5}{4}$						
1						$\frac{6}{5}$				$\frac{13}{11}$	
i" 0			0				$\frac{7}{6}$				
-1				$\frac{1}{1}$				$\frac{8}{7}$			
-2									$\frac{9}{8}$		
	-2	-1	0 i'	1	2	3	4	5	6	7	8

Inasmuch as 1 is relatively prime to all integers, we can satisfy the rule of common factors and obtain proper solutions by setting one coordinate equal to 1 and selecting the other relatively prime to i and such that both $x'i' + x''i''$ and $y'i' + y''i''$ are divisible by i. When $i = 1$,

$i' = i'' = 1$ is always a proper solution. When $i > 1$, there is always a proper solution in which $i' = 1$ and $0 < i'' < i$ or in which $i'' = 1$ and $0 < i' < i$. In either case, $i' + i'' \leq i$; and we conclude that, if i is greater than 1, at least one interval of intermediate size can always be found the sum of whose coordinates is less than or equal to i.

C. Conjoint Intervals

Two commensurable intervals are said here to be conjoint when they are more discernible than any intermediate interval. Let x'/y' be the smaller commensurable interval, let x''/y'' be the larger, and let x°/y° be any intermediate interval as in A6. In Section J of Chapter 3, $x^\circ + y^\circ$ was identified as the obscurity of the interval x°/y°. Likewise, $x' + y'$ is the obscurity of x'/y', and $x'' + y''$ is the obscurity of x''/y''. For x'/y' and x''/y'' to be conjoint, all intervals x°/y° of intermediate size must satisfy the two conditions

$$x' + y' < x^\circ + y^\circ \qquad \text{C1}$$

$$x'' + y'' < x^\circ + y^\circ \qquad \text{C2}$$

Inasmuch as x°/y° is intermediate to x'/y' and x''/y'', i' and i'' are positive, and addition of C1 multiplied by i' to C2 multiplied by i'' results in

$$(x' + y')i' + (x'' + y'')i'' < (x^\circ + y^\circ)(i' + i'') \qquad \text{C3}$$

Addition of A10 and A11 gives us

$$(x' + y')i' + (x'' + y'')i'' = (x^\circ + y^\circ)i \qquad \text{C4}$$

Since the left sides of C3 and C4 are identical,

$$(x^\circ + y^\circ)(i' + i'') > (x^\circ + y^\circ)i$$

and division by $x^\circ + y^\circ$ yields

$$i' + i'' > i \qquad \text{C5}$$

The conclusion of the preceding section shows that at least one intermediate interval always provides the contrary to this when i is greater than 1; i cannot be less than 1; therefore, C5 can be secured only by setting i equal to 1. In other words, i = 1 is necessary for x'/y' and x''/y'' to be conjoint. That it is also sufficient is proved as follows. Let i = 1; then, because of A6,

$$i' \geqq i \text{ and } i'' \geqq i$$

Adding the first of these multiplied by $x' + y'$ to the second multiplied by $x'' + y''$ gives

$$(x' + y')i' + (x'' + y'')i'' \geqq (x' + y' + x'' + y'')i \qquad \text{C6}$$

Since the left side of this is identical to the left side of C4,

$$(x' + y' + x'' + y'')i \leqq (x^\circ + y^\circ)i$$

and cancellation of the common factor i leaves

$$x' + y' + x'' + y'' \leqq x^\circ + y^\circ \qquad \text{C7}$$

C1 and C2 follow directly from this, and we conclude that x'/y' and x''/y'' are conjoint when, and only when, their determinant i equals 1.

Comparison of A2 with A1 shows that just as x'/y' is conjoint to x''/y'' when i = 1 so x°/y° is conjoint to x''/y'' when $i' = 1$. Let $i = i' = 1$; then substitution into B1 and B2 results in

$$x^\circ = x' + x''i'' \qquad \text{C8}$$

$$y^\circ = y' + y''i'' \qquad \text{C9}$$

Since $i' = 1$ here, we know from A9 that i'' increases as x°/y° increases, and it can be seen by adding C8 and C9 that $x^\circ + y^\circ$ increases as i'' increases; therefore, x°/y° becomes more obscure as it increases in size. The set of intervals conjoint to and smaller than x''/y'', then, contains one smallest interval and an unlimited number of intervals of intermediate size. As their i'' coordinate increases, they become larger, approaching x''/y'' as a limit, but their obscurity increases without limit.

Comparison of A3 with A1 shows that just as x''/y'' is conjoint to x'/y' when $i = 1$ so x°/y° is conjoint to x'/y' when $i'' = 1$. Let $i = i'' = 1$; then substitution into B1 and B2 results in

$$x^\circ = x'i' + x'' \qquad \text{C10}$$

$$y^\circ = y'i' + y'' \qquad \text{C11}$$

Since $i'' = 1$ here, we know from A9 that i' increases as x°/y° decreases, and it can be seen by adding C10 and C11 that $x^\circ + y^\circ$ increases as i' increases; therefore, x°/y° becomes more obscure as it decreases in size. The set of intervals conjoint to and larger than x'/y', then, contains one largest interval and an infinite number of intermediate intervals. As their i' coordinate increases, they become smaller, approaching x'/y' as a limit, but their obscurity increases without limit.

Inasmuch as x°/y° is conjoint to x''/y'' when $i' = 1$ and to x'/y' when $i'' = 1$, it must be conjoint to both when $i' = i'' = 1$. Let $i = i' = i'' = 1$; then substitution into B1 and B2 results in

$$x^\circ = x' + x'' \qquad \text{C12}$$

$$y^\circ = y' + y'' \qquad \text{C13}$$

which completely determines x°/y° as the one and only intermediate interval that can be conjoint to both base intervals. This presents us with three mutually conjoint intervals and shows that there cannot be more than three because, in order to have more, it would be necessary to be able to find more than one intermediate interval conjoint to both x'/y' and

x''/y''. Since this $x°/y°$ is conjoint to the base intervals, any interval intermediate to it and either base interval would be more obscure, and we see that $x°/y°$ is the most discernible interval intermediate to x'/y' and x''/y''. At most, two intervals can be conjoint to and more discernible than any given interval, in which case one is smaller and the other is larger than the given interval, and they are mutually conjoint.

Equally discernible intervals cannot be conjoint. Adding $0 = y'y'' - y'y''$ to A1 results in

$$i = y'(x'' + y'') - (x' + y')y'' \qquad \text{C14}$$

It can be verified by consulting Table I, Chapter 3, that, when x'/y' and x''/y'' are equally discernible, $x' + y' = x'' + y'' \geqq 5$ and $x'' - x' = y' - y'' \geqq 1$. Thus, when $x' + y' = x'' + y''$, C14 gives $i = (x' + y')(y' - y'') \geqq 5$, and x'/y' cannot be conjoint to x''/y''. The corollary follows that two conjoint intervals cannot be equally discernible, for, if they were, they would not be conjoint. See Plot 5.

When two intervals are not conjoint, there are always one or more intermediate intervals that are more discernible than one or both of the nonconjoint intervals. By C4,

$$x° + y° = A(i' + i'')/i \qquad \text{C15}$$

where

$$A = \frac{(x' + y')i' + (x'' + y'')i''}{i' + i''} \qquad \text{C16}$$

Here $x°/y°$ is intermediate to x'/y' and x''/y''; therefore, as in A6, i' and i'' are both positive, A is an interpolation between $x' + y'$ and $x'' + y''$, and we recognize the following alternatives:

$$x' + y' < A < x'' + y'' \qquad \text{C17}$$

$$x' + y' > A > x'' + y'' \qquad \text{C18}$$

$$x' + y' = A = x'' + y'' \qquad \text{C19}$$

$\frac{x'}{y'} = \frac{3}{2}$, $\frac{x''}{y''} = \frac{4}{1}$, $i = 5$ Plot 5.											
8				$\frac{7}{2}$							
7		$\frac{5}{1}$					$\frac{8}{3}$				
6										$\frac{9}{4}$	
5			$\frac{4}{1}$					$\frac{7}{3}$			
4						$\frac{5}{2}$					
3				$\frac{3}{1}$							
2											
1					$\frac{2}{1}$					$\frac{5}{3}$	
i'' 0			0					$\frac{3}{2}$			
-1						$\frac{1}{1}$					$\frac{4}{3}$
-2											
	-2	-1	0 i'	1	2	3	4	5	6	7	8

When x'/y' and x''/y'' are not conjoint, $i > 1$, and there is at least one intermediate interval with i' and/or i'' equal to 1 and $i' + i'' \leqq i$. Identifying $x°/y°$ with such an interval, we see by C15 that $x° + y° \leqq A$, C17 gives $x° + y° < x'' + y''$, C18 gives $x° + y° < x' + y'$, and C19 gives $x^\circ + y^\circ$ less than both $x' + y'$ and $x'' + y''$ because, as we see in Plot 5, x°/y° is conjoint to x'/y' or x''/y'' or both and conjoint intervals cannot be equally discernible.

D. The Aural Interpretation of Indiscernible Intervals

In accordance with Section J, Chapter 3, we consider an interval to be discernibly commensurable when its obscurity is less than or equal to 2N. We consider it to be indiscernible when its obscurity is greater than 2N or when its frequency ratio is incommensurable. Setting N equal to 8 limits the discernible intervals to the forty listed in Table I, Chapter 3, but many other intervals are commonly encountered in musical practice; therefore, it is important to consider the aural interpretation of the indiscernible intervals.

The smallest discernible interval is the unison, the largest is a semitone less than four octaves, and the others are distributed between these in close enough proximity to each other that any indiscernible interval of less than four octaves can be said to approximate some discernible interval. The author is of the opinion that any interval that is not discernible in itself but which approximates closely enough the size of a discernible interval partakes of the characteristic sound of the discernible interval and is actually heard as that interval or, at least, as an approximation therof. That this theory is not original with the author or new at this time is evident from the following quotations. Stumpf:

> Very small deviations of vibration frequencies from the simple ratios of the intervals do not cause a noticeable change in the degree of fusion. Thus, if the frequency of one of the tones of the fifth is slightly changed so that its vibration ratio of 2:3 is modified its degree of fusion remains that of the interval 2:3. But if the deviation is increased its fusion becomes that of its neighboring

> interval, without passing through any intermediate degrees of fusion. An interval retains its degree of fusion, then, until its vibration ratio is changed sufficiently so that it becomes a different interval, when it assumes the fusion of that interval. How rapidly this transition occurs depends upon the degree of the initial fusion. This law holds for all but the lowest degrees of fusion.[1]

Pratt:

> For a long time the approximate physical conditions for intervals have been known. Intervals are determined roughly by the ratios sustained by the frequencies. But only roughly. Just as pitch remains constant through small changes of frequency, so intervals remain unaltered through small changes of ratio. There is an interval-limen just as there is a pitch-limen, and the one can not be deduced from the other. According to recent research the pitch-limen at 40 decibels between 500 and 4000 cycles is about 5 cents. The limen for interval-discrimination is very much larger. If observers, using the method of average error, are asked to alter the variable member of a pair of tones until a given interval is reached, the m.v. of the average settings for all intervals is about 20 cents. This value must not of course be regarded as final. The judgment is not an easy one to make, the method used undoubtedly has an important bearing on the outcome, and individual differences among observers may not be negligible. More recent investigations with an entirely different method and with a larger number of observers but with only three or four intervals, gave values over 30 cents. Even this discrepancy, however, is not nearly so large as those which have appeared in studies of pitch-discrimination.[2]

---

1. As quoted by Max Schoen, The Psychology of Music, p. 49. Stumpf defined "fusion" as experiencing of two elements as one; Külpe defined it as the experience of belonging-togetherness. It depends on simplicity of vibration ratio and appears to be practically synonymous with dicernibleness.

2. Carroll C. Pratt, "Tonal Fusion," Psychological Review, vol. 41 (1934), p. 95.

Hindemith:

> Our somewhat complicated system of musical notation has the advantage of giving the singer or the player (especially of untempered instruments) in most cases a clear impression of the melodic or harmonic intentions of the composer. For analysis of the sound itself, on the other hand, it is not only worthless but actually a hindrance. For in such analysis our thesis must be that all intervals and chords are perceived, independently of their notation, as the ear first hears them, without reference to what has gone before or what comes after. The ear does not hesitate, in the course of this perception, between making all the necessary calculations of minute interval-differences, on the one hand, and, on the other, applying to each chord or interval the measurements derived from the simlest proportions of the overtone series. It always adopts the latter course, and hears every interval, even such as do not actually fit, as being of about the size of one of the intervals that we know from our two series. An interval whose tones stand only roughly in the proportions 5:6 is always heard by the ear as a minor third, whether it is written and intended by the composer as an augmented second, a minor third, or a doubly diminished fourth. Aural analysis thus takes account of no diminished or augmented intervals except the tritone; it hears all other intervals as forms of the intervals derived from the first six tones of the overtone series.[3]

Recalling that x°/y° represents only commensurable frequency ratios, and desiring a notation that can represent <u>any</u> frequency ratio, we let x equal the frequency of the higher tone and y equal that of the lower as in previous chapters. Then x/y can represent any frequency ratio, commensurable or not. If the ratio is commensurable, a pair of relatively prime integers x° and y° can always be found such that

$$x^\circ/y^\circ = x/y \qquad \text{D1}$$

---

3. Paul Hindemith, <u>The Craft of Musicl Composition</u>, Book I, pp. 93-94. The "intervals that we know from our two series" are the familiar ones of the Western musical scale.

and $$g = x/x^\circ = y/y^\circ \tag{D2}$$

Let x/y be any indiscernible interval, let x'/y' be the largest discernible interval smaller than x/y, and let x''/y'' be the smallest discernible interval larger than x/y unless x/y is larger than any discernible interval, in which case let x''/y'' be the hypothetical interval 1/0. Then

$$x'/y' < x/y < x''/y'' \tag{D3}$$

$$x' + y' \leqq 2N \tag{D4}$$

$$x'' + y'' \leqq 2N \tag{D5}$$

and, since no discernible interval is intermediate to x'/y' and x''/y'', they are conjoint and can be referred to as consecutive discernible intervals.

According to the theory advanced here, x/y must be interpreted as an approximation of either x'/y' or x''/y''. If x/y is heard as being x'/y', then x'/y' is referred to as its interpretation; if x/y is heard as an approximation of x''/y'', we call x''/y'' its interpretation. The intermediate interval may be called a tuning (or a mistuning) of the discernible interval that it approximates. As a tuning, it is subject to qualification as to accuracy, recognizability, and acceptability.

As an aid to relating Stumpf's words to our own, the author suggests that by "interval" he means what we call a discernible interval and that his term "fusion" designates a quality that we relate to the discernibleness and characteristic sound of the "interval." Thus understood, he appears to be concerned with the question as to when x/y approximates x'/y', when it approximates x''/y'', and how rapidly the interpretation of x/y changes as it passes from x'/y' to x''/y''. Certainly, the development and application of our theory depend on a precise answer to this question. Since the ear cannot judge whether an indiscernible interval is commensurable or not, what enables it to make the distinctions demanded here? In answer, we point to the characteristic sound of the interpretation, to a sense of the size of an interval (which, as Pratt observed, is not very accurate), and to the beats that are heard in a mistuned interval.

E. Beats and the Point of Division

The phenomenon of beats is of decisive importance in discerning the interpretation of a given indiscernible interval. Since x/y is intermediate in size to two consecutive discernible intervals, two rates of beating are present, but the slower beats predominate in the listener's experience. As a result, x/y is interpreted as a mistuning of the discernible interval that gives rise to the slower beats.

Let $mx + ny$ and $m'x + n'y$ be any two tones in the aural spectrum of x/y that coincide when $x/y = x'/y'$ but which differ in frequency when $x/y > x'/y'$. Then

$$mx' + ny' = m'x' + n'y'$$

or

$$(m - m')x' = (n' - n)y' \qquad \text{E1}$$

Inasmuch as $(m - m')x'$ and $(n' - n)y'$ are equal integers, they have the same factors; and, since y' is a factor of $(n' - n)y'$, it must also be a factor of $(m - m')x'$. But x' and y' are relatively prime; therefore, y' is a factor of $m - m'$. Thus $m - m' = ky'$, and, because of E1, $n' - n = kx'$, where k is the greatest common divisor of $m - m'$ and $n' - n$ as was seen to be the case in Chapter 3, Section C. The difference between $mx + ny$ and $m'x + n'y$ is $f - f' = (m - m')x - (n' - n)y$, and substitution of ky' for $m - m'$ and kx' for $n' - n$ gives

$$f - f' = k(y'x - x'y) = kg'' \quad \text{E2}$$

where $$g'' = y'x - x'y \quad \text{E3}$$

Recalling that k is a positive integer, we note that different values of k arise from considering different pairs of spectral tones in one coincidence, those lying nearest each other in the approach to a coincidence giving $k = 1$. It follows that the difference in frequency between adjacent tones in a near coincidence equals g''. Since this is the frequency difference between any pair of adjacent tones in the spectrum of x/y, then all pairs of adjacent spectral tones have the same frequency difference in any one interval. When x/y is only slightly greater than x'/y', this frequency difference gives rise to the beats that are heard in the proximity of a discernible interval, and g'' is the rate of beating.

Likewise, $$g' = yx'' - xy'' \quad \text{E4}$$

and equals the difference in frequency between tones in the spectrum of x/y that coincide when $x/y = x''/y''$ and are adjacent when $x/y < x''/y''$. When x/y is slightly less than x''/y'', g' is the rate of beating.

When $g'' < g'$, g'' predominates, E3 and E4 give

$$y'x - x'y < yx'' - xy''$$

$$(y'+ y'')x < (x'+ x'')y$$

$$x/y < (x'+ x'')/(y'+ y'') \quad \text{E5}$$

and x/y is heard as an approximation of x'/y'. When $g' < g''$, g' predominates,

$$(x'+ x'')/(y'+ y'') < x/y \quad \text{E6}$$

and x/y is heard as an approximation of x''/y''. Because of this, $(x'+x'')/(y'+ y'')$ is found to be a point in the transition from x'/y' to x''/y'' on one side of which x/y is heard as a tuning of x'/y' and on the other side of which x/y is heard as a tuning of x''/y''. In itself, $(x'+ x'')/(y'+ y'')$ is "neutral" because it may be interpreted as a tuning of either

x'/y' or x''/y'' altho it is recognizable as neither. For these reasons, we call it the <u>point of division</u> between the consecutive discernible intervals that it separates. For convenience, we let $x° = x' + x''$ and $y° = y' + y''$ as in C12 and C13; then $x°/y°$ is the point of division. Since x'/y' and x''/y'' are consecutive discernible intervals and $x°/y°$ is intermediate to them, $x°/y°$ cannot be discernible, but it is nevertheless the least obscure interval intermediate to x'/y' and x''/y''.

This information makes possible the construction of Table I, wherein all the discernible intervals and the points of division are listed in the order of increasing size. Inasmuch as N is not given one fixed value here, allowance is made for it to be either 7 or 8. Thirty-two intervals are discernible with N = 7, and these are among the forty that are discernible with N = 8. Six of the intervals that are points of division with N = 7 and two new large intervals become discernible when N is increased to 8. Three of the points of division are recognizable as being familiar intervals. The major second with frequency ratio 9/8 divides the unison from 8/7; the minor seventh 16/9 divides 7/4 from 9/5; and the perfect fourth plus two octaves 16/3 separates 5/1 from 11/2. Under ordinary circumstances, these intervals, being indiscernible, cannot be tuned directly by ear; but they can be obtained indirectly by adding and subtracting other intervals. Most of the points of division, however, are not familiar, are not easily obtained, and are of little interest in themselves.

The view formulated here agrees in part and differs in part with Hindemith, who says that the intervals 5/4, 6/5, 7/6, 9/7, 11/9, and 81/64 (the Pythagorean third, 408 cents) are all heard as thirds, that "we cannot say just where the change from a minor third to a major third takes place," and

Table I

Discernible Intervals and Points of Division

Sizes of intervals are in the first column; frequency ratios are in the columns corresponding to the value of N and the resulting classification of the intervals.

Size in cents	N = 7		N = 8	
	Disc. int.	Pt. of div.	Disc. int.	Pt. of div.
0	1/1		1/1	
204				9/8
231		8/7	8/7	
248				15/13
267	7/6		7/6	
289		13/11		13/11
316	6/5		6/5	
347		11/9		11/9
386	5/4		5/4	
418				14/11
435		9/7	9/7	
454				13/10
498	4/3		4/3	
551		11/8		11/8
583	7/5		7/5	
617		10/7		10/7
702	3/2		3/2	
782		11/7		11/7
814	8/5		8/5	
841		13/8		13/8
884	5/3		5/3	
933		12/7		12/7
969	7/4		7/4	
996		16/9		16/9
1018	9/5		9/5	

Table I, Continued

Size in cents	N = 7		N = 8	
	Disc. int.	Pt. of div.	Disc. int.	Pt. of div.
1049		11/6		11/6
1200	2/1		2/1	
1339				13/6
1365		11/5	11/5	
1382				20/9
1404	9/4		9/4	
1431		16/7		16/7
1467	7/3		7/3	
1516		12/5		12/5
1586	5/2		5/2	
1654		13/5		13/5
1698	8/3		8/3	
1729				19/7
1751		11/4	11/4	
1783				14/5
1902	3/1		3/1	
2041		13/4		13/4
2084	10/3		10/3	
2119		17/5		17/5
2169	7/2		7/2	
2218		18/5		18/5
2249	11/3		11/3	
2288		15/4		15/4
2400	4/1		4/1	
2505				17/4
2539		13/3	13/3	
2565				22/5
2604	9/2		9/2	
2667		14/3		14/3

Table I, Continued

Size in cents	N = 7		N = 8	
	Disc. int.	Pt. of div.	Disc. int.	Pt. of div.
2786	5/1		5/1	
2898		16/3		16/3
2951	11/2		11/2	
3003		17/3		17/3
3102	6/1		6/1	
3196				19/3
3241		13/2	13/2	
3284				20/3
3369	7/1		7/1	
3488		15/2		15/2
3600	8/1		8/1	
3705		17/2		17/2
3804	9/1		9/1	
3898		19/2		19/2
3986	10/1		10/1	
4071		21/2		21/2
4151	11/1		11/1	
4228		23/2		23/2
4302	12/1		12/1	
4373		25/2		25/2
4441	13/1		13/1	
4506				27/2
4569			14/1	
4630				29/2
4688			15/1	

that "in the middle space between the outside boundaries there is a field that can belong to either third, and is assigned by the ear to the major or the minor according to the harmonic or melodic context."[4] We know from the quotation in Section D that, in agreement with us, Hindemith regards 6/5 as the ideal tuning of the minor third. Doubtless he also regards 5/4 as the ideal major third and 81/64 as a deviation therefrom. The author believes, however, that we can say that the change from a minor third to a major third takes place at 11/9 and that this interval has the qualities that Hindemith ascribes to a "field in the middle space between the outside boundaries." The author says moreover that, being an independently discernible interval, 7/6 is not to be confused with the minor third and, as either a discernible interval or a point of division, 9/7 is distinguishable from the major third.

F. Nuclei

Two consecutive discernible intervals and their point of division constitute a nucleus of three mutually conjoint intervals. Let $x°/y°$ be the point of division between x'/y' and x''/y'' as in Section E. Then, because of the mutual conjointness of these three intervals,

$$i = y'x'' - x'y'' = 1 \qquad \text{F1}$$

$$i' = y°x'' - x°y'' = 1 \qquad \text{F2}$$

$$i'' = y'x° - x'y° = 1 \qquad \text{F3}$$

The intervals by which these intervals differ are as follows:

---

4. Paul Hindemith, The Craft of Musical Composition, Book I, pp. 71-72.

$$(x''/y'')/(x'/y') = (y'x'')/(x'y'') = (x'y'' + 1)/(x'y'') \quad F4$$
= the interval by which x''/y'' exceeds x'/y'.

$$(x''/y'')/(x^\circ/y^\circ) = (y^\circ x'')/(x^\circ y'') = (x^\circ y'' + 1)/(x^\circ y'') \quad F5$$
= the interval by which x''/y'' exceeds x°/y°.

$$(x^\circ/y^\circ)/(x'/y') = (y'x^\circ)/(x'y^\circ) = (x'y^\circ + 1)/(x'y^\circ) \quad F6$$
= the interval by which x°/y° exceeds x'/y'.

In the frequency ratio of each of these three latter intervals, the numerator exceeds the denominator by 1. Because of this, they are known as superparticular ratios, and the numerator and denominator are relatively prime in each interval.

Equations F4 thru F6 plainly show how the superparticular intervals are obtained from the conjoint intervals. The inverse operation is simply a matter of noting the greatest common divisors of various numerators and denominators. For example, x° is the greatest common divisor of $x^\circ y''$ and $y'x^\circ$ and y° is that of $y^\circ x''$ and $x'y^\circ$. This process yields the original conjoint intervals and no others. Thus, every nucleus proves to be unique and can be identified as well by the superparticular intervals as by the conjoint intervals.

F5 and C12 give us

$$(x''/y'')/(x^\circ/y^\circ) = 1 + 1/(x^\circ y'') = 1 + 1/(x'y'' + x''y'') \quad F7$$

F6 and C13 give us

$$(x^\circ/y^\circ)/(x'/y') = 1 + 1/(x'y^\circ) = 1 + 1/(x'y'' + x'y') \quad F8$$

and the relative values of these two intervals depend on the relative values of $x''y''$ and $x'y'$. Recalling F1 enables us to make the following observations:

When $y' \leqq y''$, then $x' < x''$, $x' + y' < x'' + y''$,
$$x'y' < x''y'', \text{ and } (x''/y'')/(x^\circ/y^\circ) < (x^\circ/y^\circ)/(x'/y') \quad F9$$

When $x' \geqq x''$, then $y' > y''$, $x' + y' > x'' + y''$,
$$x'y' > x''y'', \text{ and } (x''/y'')/(x^\circ/y^\circ) > (x^\circ/y^\circ)/(x'/y') \quad F10$$

F1 does not permit $y' > y''$ when $x' < x''$, for then

$y'x'' - x'y'' > x'' - x' \geqq 1$; therefore, F9 and F10 are the only possibilities.

This reminds us that the conjoint intervals x'/y' and x''/y'' cannot be equally discernible and demonstrates that $(x''/y'')/(x^\circ/y^\circ)$ and $(x^\circ/y^\circ)/(x'/y')$ cannot be equal, the interval by which the point of division differs from the more discernible base interval being the larger. In referring to the change in the interpretation of x/y as it passes from x'/y' to x''/y'', Stumpf says, "How rapidly this transition occurs depends upon the degree of the initial fusion." We find that the size of $(x''/y'')/(x^\circ/y^\circ)$ relative to that of $(x^\circ/y^\circ)/(x'/y')$, therefore the rapidity of the transition, does indeed depend upon the discernibleness (or fusion) of one base interval relative to that of the other.

The size in cents of any interval x/y is $M \log (x/y)$, where

$$M = 1200/\log 2 \qquad \text{F11}$$

and may be referred to as the modulus for conversion to cents. This expression for the size of x/y holds regardless of the base used for the logarithms. If the base 10 is used, $M = 3986.314$; if the base e is used, $M = 1731.234$.

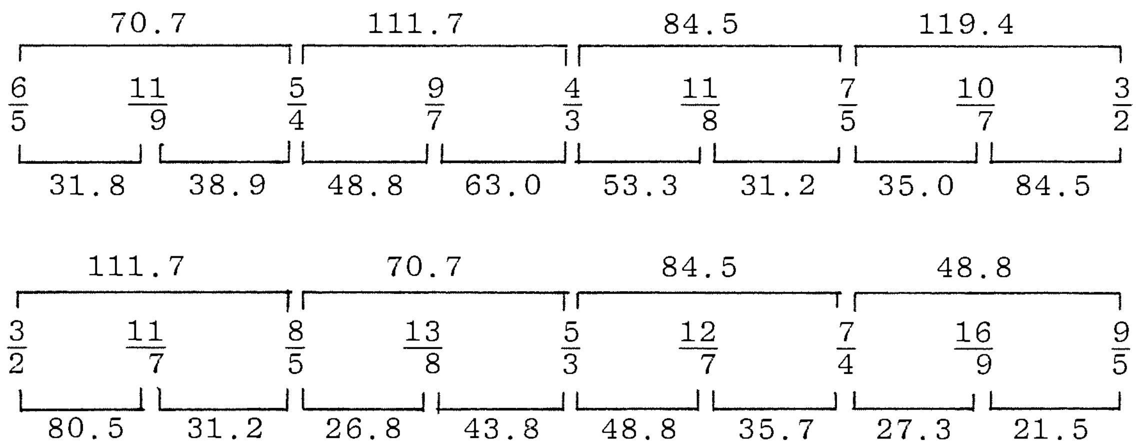
A nucleus of two consecutive discernible intervals and their point of division together with their differences in cents can be displayed in a diagram of the following type:

$M \log [(x''/y'')/(x'/y')]$

x'/y' $\qquad\qquad$ x°/y° $\qquad\qquad$ x''/y''

$M \log [(x^\circ/y^\circ)/(x'/y')]$ $\qquad$ $M \log [(x''/y'')/(x^\circ/y^\circ)]$

When actual numbers are given, successive nuclei may be linked together so as to display more than two consecutive discernible intervals with their points of division. Diagramming eight successive nuclei of Table I in this way results in



This diagram shows that the intervallic differences are by no means uniform. $(x''/y'')/(x'/y')$ varies from one nucleus to another; and, even when it is the same in different nuclei, $(x''/y'')/(x^\circ/y^\circ)$ and $(x^\circ/y^\circ)/(x'/y')$ are not the same. It is also observable that the intervallic differences between a discernible interval and the adjacent points of division are usually greater when the interval is more discernible.

In contrast to this, a plot in which the base intervals are a pair of consecutive discernible intervals (Plot 6 for example) suggests the arc tan (i''/i'), noted in connection with A9, as a measure of the intervallic differences within a nucleus that uniformly places the base intervals 90° apart and the point of division midway between them.

If $x^\circ/y^\circ = x'/y'$ as in A7, arc tan $(i''/i') = 0°$.

If $x^\circ/y^\circ = x''/y''$ as in A8, arc tan $(i''/i') = 90°$.

If x°/y° is the point of division as in F2 and F3, arc tan $(i''/i') = 45°$, which is exactly halfway between 0° and 90°.

Plot 6. $\frac{x'}{y'} = \frac{5}{3}$, $\frac{x''}{y''} = \frac{7}{4}$, $i = 1$

5										
4			$\frac{23}{13}$		$\frac{33}{19}$		$\frac{43}{25}$			
3		$\frac{11}{6}$	$\frac{16}{9}$		$\frac{26}{15}$	$\frac{31}{18}$		$\frac{41}{24}$		
2			$\frac{9}{5}$		$\frac{19}{11}$		$\frac{29}{17}$		$\frac{39}{23}$	
1			$\frac{2}{1}$	$\frac{7}{4}$	$\frac{12}{7}$	$\frac{17}{10}$	$\frac{22}{13}$	$\frac{27}{16}$	$\frac{32}{19}$	
i'' 0				0	$\frac{5}{3}$					
-1						$\frac{3}{2}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{18}{11}$	
-2							$\frac{1}{1}$		$\frac{11}{7}$	
-3										
	-3	-2	-1	0 i'	1	2	3	4	5	6

G. Recognizability of Tunings

In Chapter 3, x/y was equal to x°/y°, but here it is not necessarily so. When x°/y° was designated as the point of division, x/y was left free to be any interval. In order to relate x/y in a uniform way to the three mutually conjoint intervals of our nucleus, we adopt g''/g' in place of i''/i' and use arc tan (g''/g') as a measure of x/y's relationship to x'/y', x°/y°, and x''/y''. Dividing E3 by E4 results in

$$\frac{g''}{g'} = \frac{y'x - x'y}{yx'' - xy''} = \frac{(x/y - x'/y')y'}{(x''/y'' - x/y)y''} \qquad \text{G1}$$

which, by comparison with A9, is seen to be equvalent to i''/i' when $x/y = x°/y°$. Arc tan (g''/g') increases from 0° to 90° as x/y increases from x'/y' to x''/y'' and equals 45° when x/y matches the point of division. Let Arc denote this angle; then

$$\text{Arc} = \arc\tan(g''/g') \qquad \text{G2}$$

Arc is a measure of the recognizability of a tuning, that is, the recognizability of an indiscernible interval as an approximation of a discernible interval. The author proposes to rate the recognizability of a tuning according to the value of Arc as follows:

Value of Arc	Interpretation	Recognizability
0° to 15°	x'/y'	Good (easy)
15° to 30°	x'/y'	Fair (moderate)
30° to 45°	x'/y'	Poor (hard)
45° to 60°	x"/y"	Poor (hard)
60° to 75°	x"/y"	Fair (moderate)
75° to 90°	x"/y"	Good (easy)

Table I serves as a guide to the interpretation of any given interval. For example, let us consider the Pythagorean major sixth, with frequency ratio 27/16. Its size, 906 cents, makes it intermediate to the discernible interval 5/3 and the adjacent point of division 12/7 as can also be seen in Plot 6. Accordingly, its interpretation is 5/3, and Arc is 14 degrees. For this interval and certain other indiscernible intervals of the diatonic scale in Pythagorean tuning, Table II lists the size in cents, the frequency ratio, the base intervals, the interpretation as determined from Table I, and Arc in degrees. For most intervals, the base intervals obtained with N = 8 are the same as those obtained with N = 7. In cases in which they are not, those obtained

Table II

Interpretations and Values of Arc of Certain Intervals of the Diatonic Scale in Pythagorean Tuning

For most intervals, the base intervals obtained with N = 8 are the same as those obtained with N = 7. In cases in which they are not, those obtained with N = 7 are used. The asterisks mark the interpretations of the given intervals.

Name of interval	Size in cents	x/y	Base intervals		Arc in degrees
			x'/y'	x"/y"	
Min. 3rd	294.13	32/27	7/6	*6/5	56.31
Maj. 3rd	407.82	81/64	*5/4	4/3	17.10
Dim. 5th	588.27	1024/729	*7/5	3/2	6.97
Aug. 4th	611.73	729/512	*7/5	3/2	38.03
Min. 6th	792.18	128/81	3/2	*8/5	58.39
Maj. 6th	905.87	27/16	*5/3	7/4	14.04

with N = 7 are used. The asterisks mark the interpretations of the given intervals.

According to these values of Arc, the Pythagorean thirds and sixths rate as follows:

1. The minor intervals are hardly recognizable as tunings of the corresponding pure intervals.
2. The major third affords fair recognizability of its interpretation.
3. The major sixth is easily recognized as a tuning of its pure counterpart.

By far, the best rating goes to the diminished fifth as a tuning of 7/5, but this interval has not been favored by the theorists. It may have been confused with its inversion, the augmented fourth, which is only 23.46 cents larger and yet is a very poor tuning of the same interval.

This simple and ancient tuning is based on the principle that the octaves, perfect fifths, and perfect fourths are all pure. This results in the thirds and sixths being mistunings of their pure counterparts as was observed by Alexander Malcolm, who wrote around 1721 that "some and even the Generality ... tune not only their octaves, but also their 5ths as perfectly ... Concordant as their ear can judge, and consequently make their 4ths perfect, which indeed makes a great many Errors in the other Intervals of 3rd and 6th."[5] These "Errors" can hardly be anything other than the deviations of these intervals from their interpretations. For these particular intervals, the deviations are all 21.5 cents; but the values of Arc are all different, ranging from 14.0 to 58.4 degrees and affecting the minor intervals much more adversely than the major.

Lowering the tones A, E, and B each a syntonic comma converts the C-major diatonic scale from the Pythagorean to the just tuning. Then all of the thirds and sixths are pure except D-F and F-D, and four new indiscernible intervals appear. These are listed in Table III, where we find that the just augmented fourth is practically equal to the Pythagorean diminished fifth and the just diminished fifth differs but little from the Pythagorean augmented fourth, the difference in both cases being the schisma, 1.954 cents. This could be confusing. The large perfect fourth A-D and small perfect fifth D-A are easily recognized as tunings of the pure intervals but have always been considered unacceptable because of their dissonance.

Whatever tuning system is employed, four perfect fifths minus two octaves make a major third. In Pythagorean tuning, the fifths are pure but the third is a comma too large. In just intonation, three of the fifths and the third are pure but one of the fifths is a comma too small. In meantone temperament, the third is pure but all four of the fifths

---

5. J. Murray Barbour, Tuning and Temperament, pp. 3 and 4.

Table III

Interpretations and Values of Arc of Four Intervals of the Diatonic Scale in Just Intonation

The asterisks mark the interpretations of the given intervals.

Name of interval	Size in cents	x/y	Base intervals x'/y'	Base intervals x"/y"	Arc in degrees
Large per. 4th	519.55	27/20	*4/3	7/5	11.31
Aug. 4th	590.22	45/32	*7/5	3/2	9.46
Dim. 5th	609.78	64/45	*7/5	3/2	35.54
Small per. 5th	680.45	40/27	7/5	*3/2	84.81

are ¼ comma too small. Table IV lists the interpretations and values of Arc of ten intervals of meantone temperament. The major third and minor sixth are perfectly tuned, and seven other intervals are very easily recognizable approximations of their interpretations. On the other hand, the diminished fifth must be interpreted as a tuning of the perfect fifth, but it is hardly recognizable as such. Of the seven, four (min. 3rd, per. 4th, per. 5th, and maj. 6th) approximate familiar intervals of just intonation, and three (the augmented intervals) approximate frequency ratios involving the number 7. It is curious that musical theory did not recognize these ratios even tho meantone temperament was generally accepted for keyboard instruments for two hundred years.

Meantone temperament gives us a chromatic scale without enharmonic equivalents. Even if the scale is extended beyond twelve tones, enharmonic tones do not have the same pitch. Consequently, the augmented second is not equivalent to a minor third, the diminished fourth does not sound like a major third, and so forth. An advantage realized by this

Table IV

Interpretations and Values of Arc of Ten Intervals of Meantone Temperament

The asterisks mark the interpretations of the given intervals.

Name of interval	Size in cents	x/y	Base intervals x'/y'	x"/y"	Arc in degrees
Aug. 2nd	269.21	1.16824	*7/6	6/5	3.40
Min. 3rd	310.26	1.19628	7/6	*6/5	84.02
Maj. 3rd	386.31	1.25	*5/4		0
Per. 4th	503.42	1.33748	*4/3	7/5	2.28
Aug. 4th	579.47	1.39754	4/3	*7/5	86.35
Dim. 5th	620.53	1.43108	7/5	*3/2	48.43
Per. 5th	696.58	1.49535	7/5	*3/2	88.88
Min. 6th	813.69	1.6	*8/5		0
Maj. 6th	889.74	1.67185	*5/3	7/4	2.85
Aug. 6th	965.78	1.74693	5/3	*7/4	87.08

temperament is the harmonious sound of the chords of the augmented sixth such as E♭GB♭C# and B♭DFG#.

Table V lists interpretations and values of Arc of twelve intervals of the equally tempered scale. The thirds and sixths of this tuning rate much better than those of the Pythagorean tuning but worse than those of meantone temperament, the minor intervals being moderately recognizable as 6/5 and 8/5 and the major intervals being easily recognizable as their interpretations. The augmented fourth and the diminished fifth have been merged into one interval that is only fairly recognizable as 7/5. The deviation of the perfect fourth and fifth is only 1.955 cents in magnitude, which can safely be identified with the schisma and makes them highly accurate tunings of the pure intervals.

Equal temperament gives us a chromatic scale with a multiplicity of names for every tone and consequently enharmonic equivalents for every interval. This multiplicity

Table V

Interpretations and Values of Arc of Twelve Intervals of the Equally Tempered Scale

For most intervals, the base intervals obtained with N = 8 are the same as those obtained with N = 7. In cases in which they are not, those obtained with N = 7 are used. The asterisks mark the interpretations of the given intervals.

Name of interval	Size in cents	x/y	Base intervals x'/y'	Base intervals x"/y"	Arc in degrees
Min. 2nd	100	1.05946	*1/1	7/6	5.28
Maj. 2nd	200	1.12246	*1/1	7/6	24.78
Min. 3rd	300	1.18921	7/6	*6/5	68.25
Maj. 3rd	400	1.25992	*5/4	4/3	10.21
Per. 4th	500	1.33484	*4/3	7/5	.79
Dim. 5th	600	1.41421	*7/5	3/2	22.50
Per. 5th	700	1.49831	7/5	*3/2	89.61
Min. 6th	800	1.58740	3/2	*8/5	70.18
Maj. 6th	900	1.68179	*5/3	7/4	9.44
Min. 7th	1000	1.78180	7/4	*9/5	54.41
Maj. 7th	1100	1.88775	9/5	*2/1	75.65
Octave	1200	2.	*2/1		0

enables the musician to make notational distinctions proper to the Pythagorean or the meantone system even when they do not affect the actual sound.

So far in this section, we have been concerned with determining the recognizability of a given tuning. Let us now give brief consideration to determining the largest and smallest intervals that have a given degree of recognizability. From G2,

$$g''/g' = \tan \text{Arc} \qquad \text{G3}$$

and, from E3, E4, and F1,

$$x'g' + x''g'' = x \qquad \text{G4}$$

$$y'g' + y''g'' = y \qquad \text{G5}$$

Division of G4 by G5 leads to

$$\frac{x}{y} = \frac{x'g' + x''g''}{y'g' + y''g''} = \frac{x' + x''g''/g'}{y' + y''g''/g'}$$

$$= \frac{x' + x'' \tan \text{Arc}}{y' + y'' \tan \text{Arc}} \qquad \text{G6}$$

This equation gives the largest interval that has a given recognizability when $0° < \text{Arc} < 45°$ and x'/y' is the given discernible interval. It gives the smallest such interval when $45° < \text{Arc} < 90°$ and x''/y'' is the same discernible interval. For example, let us determine the largest and smallest intervals that are easily recognizable as tunings of 3/2. We set Arc = 15°, $x' = 3$, $y' = 2$, $x'' = 8$, and $y'' = 5$. Then G6 yields as the largest interval $x/y = 1.54$ or $M \log (x/y) = 748$ cents. To get the smallest interval, we set Arc = 75°, $x' = 7$, $y' = 5$, $x'' = 3$, and $y'' = 2$. Then G6 yields $x/y = 1.46$ or $M \log (x/y) = 655$ cents. Any interval between these two in size should be easily recognizable as a tuning of 3/2.

H. Pseudo Nuclei

Let x'/y' and x''/y'' be conjoint to $x°/y°$ and possibly but not necessarily conjoint to each other. Then $i \geqq 1$, $i' = i'' = 1$, B1 and B2 become

$$x° = (x' + x'')/i \qquad \text{H1}$$

$$y° = (y' + y'')/i \qquad \text{H2}$$

and addition of these two equations results in

$$x° + y° = (x' + y' + x'' + y'')/i \qquad \text{H3}$$

When $i = 1$ as in Plot 1, $x° + y°$ is greater than either $x' + y'$ or $x'' + y''$, and these intervals constitute a true nucleus of three mutually conjoint intervals as in Section F. When $i = 2$ as in Plot 2, x'/y' and x''/y'' are not conjoint, but they still cannot be equally discernible, and $x° + y°$

has a value between $x' + y'$ and $x'' + y''$. When $i > 2$ as in Plot 3, $x° + y°$ lies between or is less than $x' + y'$ and $x'' + y''$. x'/y' and x''/y'' are already known as the base intervals, $x°/y°$ will be called the apex, and all three may be referred to collectively as "the conjoint intervals" even tho the base intervals may not actually be conjoint. When $i > 1$, these intervals constitute what we call a pseudo nucleus here, because the relationships therein are similar to yet different from those in the true nucleus.

Inasmuch as $x°/y°$ is conjoint to both base intervals, all intervals intermediate in size to $x°/y°$ and either base interval are more obscure than $x°/y°$. It follows that $x°/y°$ is the most discernible interval intermediate to x'/y' and x''/y'' regardless of whether the nucleus is true or pseudo. In the preceding three sections, $x°/y°$ was the point of division between consecutive discernible intervals, and the nucleus was true. Alternately, the apex can represent a discernible interval, in which case x'/y' and x''/y'' are not consecutive discernible intervals and may or may not be conjoint.

The intervals by which the "conjoint" intervals differ are as follows:

$$(x''/y'')/(x'/y') = (y'x'')/(x'y'') = (x'y'' + i)/(x'y'') \qquad \text{H4}$$

= the interval by which x''/y'' exceeds x'/y' and is either not a superparticular ratio or must be reduced to lowest terms to be recognized as such when $i > 1$. As in F5 and F6,

$$(x''/y'')/(x°/y°) = (y°x'')/(x°y'') = (x°y'' + 1)/(x°y'') \qquad \text{H5}$$

= the interval by which x''/y'' exceeds $x°/y°$.

$$(x°/y°)/(x'/y') = (y'x°)/(x'y°) = (x'y° + 1)/(x'y°) \qquad \text{H6}$$

= the interval by which $x°/y°$ exceeds x'/y'.

The frequency ratios of these two latter intervals are always superparticular and cannot be reduced to lower terms, but all three are loosely called here "the superparticular intervals" regardless of the value of i.

The conjoint intervals completely determine the superparticular intervals whether the nucleus is true or pseudo, but the superparticular intervals do not uniquely determine the conjoint intervals when pseudo nuclei are permitted. Let the three superparticular intervals of a nucleus be given, let a be the greatest common divisor of x' and x'', and let b be that of y' and y''. Then it is evident from A1 that a and b are factors of i and that a = b = 1 when i = 1. Whatever the value of i, bx° is the greatest common divisor of $x^\circ y''$ and $y'x^\circ$, and ay° is that of $y^\circ x''$ and $x'y^\circ$. Once ay° and bx° have been picked out, the choice of a and b must be guided by the relations

$$1 \leqq a \leqq ay^\circ \qquad \text{H7}$$

$$1 \leqq b < bx^\circ \qquad \text{H8}$$

Choosing a and b immediately determines x° and y°, and dividing appropriate terms of the given superparticular intervals by x° or y° yields the terms of the base intervals. It will be found that ab is the greatest common divisor of y'x'' and x'y'' and may be used to reduce their ratio to lowest terms.

Examples of different nuclei with identical superparticular intervals are given in Table VI. The superparticular intervals of the true nucleus No. 1 are duplicated by those of the pseudo nucleus No. 2 -- evident inasmuch as 45/40 = 9/8. The superparticular intervals of the true nucleus No. 3 are duplicated by those of the pseudo nuclei Nos. 4 thru 7. In the pseudo nuclei Nos. 8 thru 10, y'x''/x'y'' is not superparticular. In Nos. 11 thru 13, y'x''/x'y'' is recognizable as superparticular when reduced to

Table VI

Different Nuclei with Identical Superparticular Intervals

The superparticular intervals are the same in Nos. 1 and 2, in Nos. 3 thru 7, in Nos. 8 thru 10, and in Nos. 11 thru 13.

No.	a	b	i	Conjoint intervals			Superparticular intervals		
				x'/y'	x°/y°	x"/y"	y'x°/x'y°	y'x"/x'y"	y°x"/x°y"
1	1	1	1	4/3	7/5	3/2	21/20	9/8	15/14
2	5	1	5	20/3	7/1	15/2	21/20	45/40	15/14
3	1	1	1	5/2	8/3	3/1	16/15	6/5	9/8
4	1	2	2	5/4	4/3	3/2	16/15	12/10	9/8
5	3	1	3	15/2	8/1	9/1	16/15	18/15	9/8
6	3	2	6	15/4	4/1	9/2	16/15	36/30	9/8
7	3	4	12	15/8	2/1	9/4	16/15	72/60	9/8
8	1	1	2	5/3	7/4	9/5	21/20	27/25	36/35
9	2	1	4	10/3	7/2	18/5	21/20	54/50	36/35
10	4	1	8	20/3	7/1	36/5	21/20	108/100	36/35
11	3	1	3	6/5	5/4	9/7	25/24	45/42	36/35
12	4	1	4	8/5	5/3	12/7	25/24	60/56	36/35
13	6	1	6	12/5	5/2	18/7	25/24	90/84	36/35

15/14, but these are pseudo nuclei nevertheless because $i > 1$.

Since H5 and H6 are small intervals, the approximation B4 of Chapter 2 may be applied to them; and, since they are superparticular ratios, the results are surprisingly simple:

$$\ln[(x''/y'')/(x°/y°)] \cong 2/(y°x'' + x°y'') \qquad \text{H9}$$

$$\ln[(x°/y°)/(x'/y')] \cong 2/(y'x° + x'y°) \qquad \text{H10}$$

Inasmuch as superparticular ratios cannot be reduced to lower terms, $y°x'' + x°y''$ is the obscurity of H5, $y'x° + x'y°$ is that of H6, and we can say that the sizes of these intervals (and indeed the sizes of all small superparticular intervals) are reciprocally proportional to their obscurities.

Division of H9 by H10 gives

$$\frac{M \log [(x''/y'')/(x^\circ/y^\circ)]}{M \log [(x^\circ/y^\circ)/(x'/y')]} \cong \frac{y^\circ x' + x^\circ y'}{y^\circ x'' + x^\circ y''} \qquad \text{H11}$$

but
$$\frac{y^\circ x' + x^\circ y'}{y^\circ x'' + x^\circ y''} \cong \frac{x' + y'}{x'' + y''} \qquad \text{H12}$$

when $(x^\circ - y^\circ)i$ is small

for $$\frac{y^\circ x' + x^\circ y'}{y^\circ x'' + x^\circ y''} \div \frac{x' + y'}{x'' + y''} = 1 + \frac{(x^\circ - y^\circ)i}{(y^\circ x'' + x^\circ y'')(x' + y')} \qquad \text{H13}$$

and it is seen that, to a limited degree of accuracy, the intervallic differences between the apex and the base intervals of a nucleus are reciprocally proportional to the obscurities of the base intervals.

I. Accuracy of Tunings

If x/y is not an exact tuning of its interpretation, it may be said to deviate therefrom. Let Q be the frequency ratio of the interval of deviation. Then Q equals x/y divided by the frequency ratio of its interpretation. If it is a tuning of x'/y',

$$Q = (x/y)/(x'/y') = (y'x)/(x'y) \qquad \text{I1}$$

and substitution into this from G6 gives

$$Q = \frac{y'(x' + x'' \tan \text{Arc})}{x'(y' + y'' \tan \text{Arc})} \qquad \text{I2}$$

whence $1 \leqq Q \leqq (y'x^\circ)/(x'y^\circ)$

when $0 \leqq \text{Arc} \leqq 45^\circ$.

If x/y is a tuning of x"/y",

$$Q = (x/y)/(x''/y'') = (y''x)/(x''y) \qquad \text{I3}$$

and substitution into this from G6 gives

$$Q = \frac{y''(x' + x'' \tan \text{Arc})}{x''(y' + y'' \tan \text{Arc})} \qquad \text{I4}$$

whence $(y''x^\circ)/(x''y^\circ) \leqq Q \leqq 1$

when $45^\circ \leqq \mathrm{Arc} \leqq 90^\circ$.

If x°/y° is a discernible interval (the interpretation of x/y) and x'/y' and x''/y'' are the adjacent points of division,

$$Q = (x/y)/(x^\circ/y^\circ) = (y^\circ x)/(x^\circ y) \qquad \text{I5}$$

and $$(x'/y')/(x^\circ/y^\circ) \leqq Q \leqq (x''/y'')/(x^\circ/y^\circ) \qquad \text{I6}$$

G6 is not applicable in this case, but it is still true that

$$1 \leqq Q \text{ when } 0 \leqq \mathrm{Arc} \leqq 45^\circ$$

and $$Q \leqq 1 \text{ when } 45^\circ \leqq \mathrm{Arc} \leqq 90^\circ.$$

When $Q < 1$, $M \log Q < 0$ but still gives the proper size (or amount) of the deviation, for

$$\log Q = -\log (1/Q) \qquad \text{I7}$$

In other words, the magnitude of the deviation is given by the absolute value of $M \log Q$, and the direction of the deviation is given by the sign of $M \log Q$. The deviation is an accepted measure of the accuracy of a tuning, the tuning being exact when $Q = 1$ (and $M \log Q = 0$).

The accuracy required to provide good recognizability of a tuning varies according to the interval being tuned, but it can be easily determined by I2 and I4. If $x'/y' < x/y < x^\circ/y^\circ$, I2 is used with Arc = 15°. If $x^\circ/y^\circ < x/y < x''/y''$, I4 is used with Arc = 75°. Table VII lists tunings and deviations thus obtained for discernible intervals from the unison to the octave in size. It is to be observed that in general the deviations are greater for the more discernible intervals and less for the more obscure ones. The deviation of least magnitude is -8.51 cents for 9/5 when Arc = 75°. It follows then that recognition will be easy for any deviation of less than 8.5 cents in magnitude.

Table VII

Tunings and Deviations that Correspond to the Limits of Easy Recognition

Disc. int.	Arc	M log (x/y)	M log Q	Disc. int.	Arc	M log (x/y)	M log Q
1/1	0°	0	0		75°	655.03	-46.93
	15°	169.33	169.33	3/2	0°	701.96	0
					15°	747.65	45.69
	75°	256.27	-10.61				
7/6	0°	266.87	0		75°	803.18	-10.51
	15°	275.88	9.01	8/5	0°	813.69	0
					15°	823.65	9.96
	75°	303.90	-11.74				
6/5	0°	315.64	0		75°	862.85	-21.51
	15°	328.33	12.69	5/3	0°	884.36	0
					15°	907.00	22.64
	75°	368.85	-17.46				
5/4	0°	386.31	0		75°	954.98	-13.85
	15°	405.52	19.21	7/4	0°	968.83	0
					15°	981.19	12.37
	75°	469.33	-28.72				
4/3	0°	498.04	0		75°	1009.09	-8.51
	15°	524.56	26.52	9/5	0°	1017.60	0
					15°	1027.35	9.76
	75°	571.06	-11.46				
7/5	0°	582.51	0		75°	1097.92	-102.08
	15°	594.44	11.93	2/1	0°	1200.00	0

J. Acceptability of Tunings

Recognizability of a tuning does not of itself confer acceptance. When tuned to the frequency ratio 3/2, the perfect fifth has always been considered to be consonant; but, as we have already noted, an interval that is smaller than 3/2 by a syntonic comma is considered to be dissonant and is not accepted as a substitute for the consonant fifth. Even more objectionable is the meantone "wolf fifth," actually the diminished sixth G#-E♭, which is 35.7 cents larger than the pure perfect fifth. The major third is universally found to be harmonious in just intonation and harsh in Pythagorean tuning. It has long been recognized that this sensation of dissonance or harshness results from the beats that are heard in an approximation of a discernible interval. The acceptability of a tuning (that is, the acceptability of an indiscernible interval as a tuning of a discernible interval), then, depends on smoothness (that is, freedom from disturbing beats) as well as recognizability.

The tones that beat in an approximation of a discernible interval are the very tones that coincide in the spectrum of the interval that is approximated. When the frequency ratio of a discernible interval is altered slightly, these tones separate to form a near coincidence of adjacent spectral tones. Such a near coincidence is called here an adjacency. We recognize two adjacencies as being the principal sources of dissonance. First is the adjacency corresponding to the lowest coincidence of two partials presented in B5 of Chapter 2. Second is the adjacency corresponding to the highest primary coincidence in the spectrum of the discernible interval.

As was observed in Chapter 3, Section G, the coefficients of the tones in the lowest coincidence or adjacency of two partials are

$$m' = 0, \; n' = x^\circ = G' \qquad \text{J1}$$

$$m = y^\circ, \; n = 0 \qquad \text{J2}$$

The frequencies of these two tones are

$$f = y^\circ x, \; f' = x^\circ y \qquad \text{J3}$$

and their frequency ratio is

$$f/f' = (y^\circ x)/(x^\circ y) = Q \qquad \text{J4}$$

which is the interval of deviation of x/y from its interpretation x°/y°. Since $x^\circ = G'$, the tone f' is of doubtful significance when $x^\circ > N$.

According to G12, G8, and G9 of Chapter 3, the coefficients of the tones in the highest primary coincidence or adjacency are

$$m' = 0,\ n' = G' = (x^\circ + y^\circ + \mu)/2 \qquad \text{J5}$$

$$m = y^\circ,\ n = G' - x^\circ \leqq 0 \qquad \text{J6}$$

The frequencies of these tones are

$$f = y^\circ x + (G' - x^\circ)y \qquad \text{J7}$$

$$= G'y + (Q - 1)x^\circ y \qquad \text{J8}$$

$$f' = G'y \qquad \text{J9}$$

and their frequency ratio is

$$R = f/f' = 1 + (Q - 1)x^\circ/G' \qquad \text{J10}$$

When x°/y° is a unison or superparticular interval, $\mu = x^\circ - y^\circ$, $G' = x^\circ$, and $R = Q$. Otherwise, $\mu < x^\circ - y^\circ$, $G' < x^\circ$, and $R \neq Q$ unless $Q = 1$. When $x/y > x^\circ/y^\circ$, $Q > 1$ and $R \geqq Q$. When $x/y < x^\circ/y^\circ$, $Q < 1$ and $R \leqq Q$.

Transforming J10 to

$$R - 1 = (Q - 1)x^\circ/G' \qquad \text{J11}$$

helps us to understand the relationship between the two adjacencies whose frequency ratios are Q and R. Under the present circumstances, Q and R are approximately equal to 1,

$$Q - 1 \cong \ln Q \qquad \text{J12}$$

and

$$R - 1 \cong \ln R \qquad \text{J13}$$

Substitution from these equations into J11 results in

$$\ln R \cong (\ln Q)x^\circ/G' \qquad \text{J14}$$

which, upon being multiplied by M, becomes

$$M \log R \cong (M \log Q)x^\circ/G' \qquad \text{J15}$$

Altho approximate, this equation is truly indicative of the real relationship.

It was found in Chapter 2, Section B, that the roughness of the beats depends on the number of cycles per beat, the center frequency of the beating tones, and the intensity of the beats. Let

$$F = \tfrac{1}{2}(f + f') \qquad \text{J16}$$

= center frequency of the adjacency in cps,

and let Δt be the period of one beat as in Chapter 2. Then the number of cycles per beat is

$$F\Delta t = \tfrac{1}{2}(f + f')/|f - f'| \qquad \text{J17}$$

$$= \tfrac{1}{2}(f/f' + 1)/|f/f' - 1| \qquad \text{J18}$$

Substitution from J4 into this gives us the number of cycles per beat for the lowest adjacency of two partials:

$$F\Delta t = \tfrac{1}{2}(Q + 1)/|Q - 1| \qquad \text{J19}$$

$$\cong 1/|\ln Q| = 1731/|M \log Q| \qquad \text{J20}$$

Substitution from J10 into J18 gives us the number of cycles per beat for the highest primary adjacency:

$$F\Delta t = \tfrac{1}{2}(R + 1)/|R - 1| \qquad \text{J21}$$

$$\cong 1/|\ln R| = 1731/|M \log R| \qquad \text{J22}$$

In order for these adjacencies to produce the sensation of beats, $F\Delta t$ must be more than about 7.5. By J20 and J22, this requirement is satisfied when $|M \log Q|$ and $|M \log R|$ are less than 231 cents. As long as Q is the deviation of x/y from its interpretation, it cannot be greater than the interval between the interpretation and the applicable point of division. As can be seen in Table I, the three largest differences between a discernible interval and an adjacent point of division are 231 cents between 1/1 and 8/7, 165 cents between 2/1 and 11/5, and 151 cents between 2/1 and 11/6. This means that $|M \log Q| \leqq 231$ cents as required for definite beats. For the unison and the superparticular intervals, R = Q and therefore $|M \log R| \leqq 231$ cents. For the perfect twelfth (3/1), $|M \log Q| \leqq 151$ cnts and, by J15, $|M \log R| \leqq 226$ cents. Upon checking other intervals, it is found that Q and R are always within the size limits necessary to produce definite beats.

As the deviation decreases from 231 cents to somewhere near 30 cents, the number of cycles per beat increases from 7.5 to about 60, and the beats become rougher. For smaller deviations, the beats are not noticeably rougher. In fact, as the deviation drops below 20 cents, the beats become more tolerable and the tuning becomes more acceptable.

The center frequency is not inherent in the size of an interval. Transposing an interval changes the center frequency and therefore the roughness of the beats; consequently, beats due to mistuning may be objectionable when an interval is in one range (or octave) of the musical scale and acceptable when it is in another range. However, a tuning that is accepted in the middle range is at least tolerated in lower and higher ranges.

In Chapter 2, Section A, I_r and I_s are the respective intensities of tones r and s of a spectrum, and $2\sqrt{I_r I_s}$ is identified as a measure of the intensity of the beats that are present when there is a small interval between the tones. Let I and I' be the respective intensities of tones f and f', and let I" denote the intensity of the beats. Then, according to this measure,

$$I'' = 2\sqrt{II'} \qquad \text{J23}$$

and, by F12 of Chapter 1, the intensity level of the beats is

$$IL'' = 10 \log_{10} (2\sqrt{II'}) + 100$$

$$IL'' = 10\,[\log 2 + \tfrac{1}{2}(\log I + \log I')] + 100$$
$$= 10 \log 2 + \tfrac{1}{2}(10 \log I + 100 + 10 \log I' + 100)$$
$$= 3 + \tfrac{1}{2}\,(IL + IL') \qquad J24$$

where IL and IL' are the respective intensity levels of tones f and f'.

When f and f' are the frequencies of partials, the indexes G and G' are equal to the numbers of the partials, and F1 and F2 of Chapter 2 become applicable when rewritten in the form

$$IL = IL_N + 6(N - G) \qquad J25$$
$$IL' = IL_N + 6(N - G') \qquad J26$$

These formulas relate not only to the intensity levels of partials of typical musical tones but also to the approximate loudnesses of other spectral tones when the definition of N is broadened to denote the maximum index of an audibly significant spectral tone as in Chapter 3, Section F. Substitution from J25 and J26 into J24 results in

$$IL'' = 3 + \tfrac{1}{2}[2IL_N + 6(2N - G - G')]$$
$$= 3 + IL_N + 3[2N - (G + G')]$$
$$= IL_N + 3[2N + 1 - (G + G')] \qquad J27$$

J1 and J2 apply to the lowest adjacency of two partials; and they give us $G = y°$, $G' = x°$, and $G + G' = x° + y°$. J6 applies to the highest primary adjacency; and it gives us $G = m - n = y° - G' + x°$ or, again, $G + G' = x° + y°$. Making this substitution into J27 results in

$$IL'' = IL_N + 3[2N + 1 - (x° + y°)] \qquad J28$$

This last equation shows clearly that the intensity, hence the roughness, of the beats is greater when the approximated interval is more discernible; but a tuning becomes more acceptable when accuracy improves below 20 cents; therefore, the more discernible intervals must be tuned more accurately so as to diminish the roughness of the beats.

Rough approximations of the most discernible intervals (unison and octave) are therefore the most dissonant, and these intervals are customarily tuned as accurately as possible. The accuracy of tuning required for other intervals is perhaps best judged by the ear of the musician as reflected in the historical record.

The perfect fifth and perfect fourth in equal temperament with $|M \log Q| = 2.0$ cents and in meantone temperament with $|M \log Q| = 5.4$ cents were accepted, whereas the small per. 5th and the large per. 4th in just intonation with $|M \log Q| = 21.5$ cents were rejected in spite of easy recognition. The meantone maj. 6th with $M \log Q = 5.4$ cents was agreeable, but the Pythagorean maj. 6th and maj. 3rd with $M \log Q = 21.5$ cents were very objectionable. The author thinks the equally tempered maj. 6th with $M \log Q = 15.6$ cents and maj. 3rd with $M \log Q = 13.7$ cents are merely tolerated. The meantone min. 3rd with $M \log Q = -5.4$ cents was easily accepted, but the equally tempered min. 3rd with $M \log Q = -15.6$ cents and min. 6th with $M \log Q = -13.7$ cents are not only inaccurately tuned but also not easily recognized. These two cannot be rated as good tunings, but the relative obscurity of their interpretations results in weak beats and therfore little roughness.

Meantone temperament provided not only exactly tuned major thirds but also diminished fourths, whose interpretations were also 5/4 with Arc = 37° (poor recognizability). The deviation of these intervals, 41 cents, was known as "the wolf" and certainly made them unacceptable as tunings of a major third. The major third (or dim. 4th) of equal temperament has just 1/3 the deviation of the meantone dim. 4th. In a book he published in 1739, van Blankenburg referred to these deviations as "young wolves, each 1/3 of the large wolf."[6]

---

6. J. Murray Barbour, Tuning and Temperament, p. 120.

K. Consonance and Dissonance

Consonance and dissonance are regarded here as intrinsic qualities of harmonic intervals, fixed by the frequency ratio (or size) of an interval and not affected by high or low position in the scale. On the other hand, the comparable qualities of smoothness and roughness vary with the range in which an interval is located. It is thus possible for consonant intervals to be rough in the low bass range (below 3C) and for dissonant intervals to be smooth in the high treble range. In the middle range, however, dissonance and roughness are practically the same.

An interval exhibits the quality of dissonance when, and only when, it is close in size to a sufficiently discernible interval. This happens not only when the dissonant interval is indiscernible and the proximate discernible interval is its interpretation but also when the interval sounded is a point of division or even another discernible interval. An interval is consonant when it is not dissonant, that is, when the proximate discernible intervals are not discernible enough or not near enough in size to the interval sounded to create objectionable beats. The most consonant intervals, then, are the unison and the octave. The discernible intervals that are nearest the unison in size are those whose frequency ratios are 7/6 and possibly 8/7, but they are among the least discernible intervals and differ from the unison by 267 and 231 cents respectively. The discernible intervals nearest the octave in size are 9/5, 9/4, and possibly 11/5; but they, too, are among the least discernible intervals and differ from the octave by 182, 204, and 165 cents. In contrast, the most dissonant intervals are those that approximate the most discernible intervals. Thus the major and minor seconds, the major seventh, and the minor ninth are the most dissonant familiar intervals because of their proximities to the unison and the octave.

Hence proximate discernible intervals can truly be regarded as potential sources of dissonance in the interval sounded, and we can say with Helmholtz[7] that discernible intervals "disturb" other intervals that approximate them in size. In general, not all proximate discernible intervals contribute equally to the dissonance of x/y, the interval sounded, and it is important to know which are responsible for the greatest or predominant disturbance. The two most disturbing intervals are easily related to x/y when regarded as base intervals of a nucleus, and there are two cases corresponding to whether x/y is or is not an acceptable tuning of a discernible interval.

If x/y is <u>not</u> an acceptable tuning of a discernible interval, let x'/y' and x"/y" be consecutive discernible intervals respectively smaller and larger than x/y as in D3. Being conjoint, they are base intervals of a true nucleus, and both are potential sources of dissonance in the interval sounded. If also x/y is not the point of division, one of the base intervals is its interpretation, and the other is its adjunct. If x/y is easily or fairly recognizable but not acceptable as a tuning of its interpretation, it is disturbed more by the interpretation than by the adjunct. If x/y approximates the point of division and therfore is hardly recognizable as a tuning of its interpretation, it is disturbed almost equally by both base intervals. Under these conditions, the disturbance is greater when the magnitude of the deviation $|M \log Q|$ is less and when the interpretation is more discernible.

That x'/y' and x"/y" are the proximate discernible

---

7. <u>Sensations of Tone</u>, pp. 186-187.

intervals most disturbing to x/y can hardly be doubted if it is true that the intervals that give rise to the slower beats contribute more to the dissonance of x/y. The intervals intermediate to x'/y' and x''/y'' are not discernible and therefore cannot disturb x/y. To be as disturbing as x'/y' if possible, any smaller interval would have to be conjoint to x'/y' so that no intermediate interval could be more discernible, and it would have to be more discernible than x'/y' because it would differ more in size from x/y. Such an interval is $(x'i' - x'')/(y'i' - y'')$, where i' is a small positive integer, usually 1 or 2. If it does not differ too greatly in size from x/y, it will induce beats at the rate of $(y'i' - y'')x - (x'i' - x'')y$ per second. In terms of g'' and g' as defined in E3 and E4, this equals $g''i' + g'$, which is greater than either g'' or g' as long as $x'/y' < x/y < x''/y''$. To produce a disturbance comparable to that of x''/y'', any larger interval should be conjoint to and more discernible than x''/y''. Such an interval is $(x''i'' - x')/(y''i'' - y')$, where i'' is a small positive integer, usually 1 or 2. If it does not differ too greatly in size from x/y, it will induce beats at the rate of $(x''i'' - x')y - (y''i'' - y')x$ per second. In terms of g' and g'', this equals $g'i'' + g''$, which is greater than either g' or g'' as long as $x'/y' < x/y < x''/y''$. Relations like these between the intervals and the rates of beating are illustrated in Figure 1.

There it is easily seen that, since x/y is intermediate in size to the two base intervals, any beats induced by the "outside" intervals $(x'i' - x'')/(y'i' - y'')$ and $(x''i'' - x')/(y''i'' - y')$ would be faster and therefore probably less disturbing than those resulting from x'/y' and x''/y''. It is not thought, however, that beats induced by outside discernible intervals are without effect. If, as in Figure 1, x'/y' is more discernible than x''/y'', and if

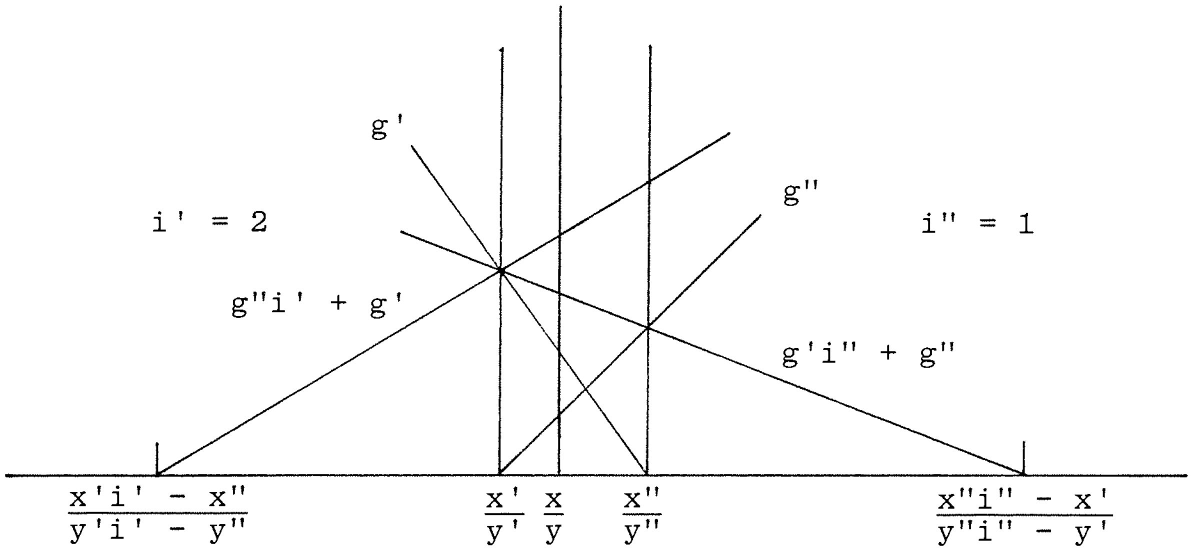


Figure 1. Relative rates of beating induced by discernible intervals that are approximated by x/y. g" is the rate of beating induced by x'/y', g' is that due to x"/y", and so forth. Here x'/y' is more discernible than x"/y".

y' > 1, then i" = 1, i' > 1, and (x"i" - x')/(y"i" - y') is conjoint to x'/y' as well as x"/y" and induces slower and more disturbing beats than (x'i' - x")/(y'i' - y"). If, on the other hand, x"/y" is more discernible than x'/y', then i' = 1, i" > 1, and (x'i' - x")/(y'i' - y") is conjoint to x"/y" as well as x'/y' and induces the slower and more disturbing beats. In summary, we can say that the outside interval conjoint to and more discernible than the less discernible base interval is conjoint to both base intervals, and the beats that it induces are slower and probably more disturbing to x/y than those due to the other outside interval and may have a marked effect on the dissonance of x/y.

Table VIII lists some indiscernible intervals that deviate from their interpretations by a comma and are therefore unacceptable tunings thereof. With each of these it also lists the base intervals respectively smaller and

Table VIII

Certain Indiscernible Intervals and the Proximate Discernible Intervals that Disturb Them

P denotes Pythagorean tuning; J, just intonation; and E, equal temperament. The asterisks mark the interpretations of the given unacceptable tunings.

Name of interval	$\frac{x}{y}$	$\frac{x'i'-x''}{y'i'-y''}$	Base intervals		$\frac{x''i''-x'}{y''i''-y'}$	Arc in degrees	M log Q in cents
			x'/y'	x"/y"			
P min. 6th	128/81		3/2	*8/5	5/3	58.4	-21.5
P min. 3rd	32/27		7/6	*6/5	5/4	56.3	-21.5
P maj. 3rd	81/64		*5/4	4/3		17.1	21.5
P maj. 6th	27/16	3/2	*5/3	7/4		14.0	21.5
J lrg. 4th	27/20		*4/3	7/5	3/2	11.3	21.5
J sml. 5th	40/27	4/3	7/5	*3/2		84.8	-21.5
P min. 7th	16/9	5/3	7/4	9/5	2/1	45.0	
E min. 6th	1.5874		3/2	*8/5	5/3	70.2	-13.7
E min. 3rd	1.1892		7/6	*6/5	5/4	68.2	-15.6
E dim. 5th	1.4142	4/3	*7/5	3/2		22.5	17.5

larger than the named intervals, and the outside discernible intervals that are near enough in size to the given indiscernible intervals to affect their consonance or dissonance. The Pythagorean major third and major sixth and the just large fourth and small fifth are not hard to recognize as tunings of their interpretations. It follows that their dissonance is traceable primarily to their interpretations; and, since the deviations are of equal magnitude, the intervals whose interpretations are more discernible are the more dissonant. They are listed in order of increasing discernibleness of their interpretations therefore in order of increasing dissonance. The influence of the outside intervals is minor; but, even so, it is such as to increase the comparative dissonance of the fourth and fifth.

Since the Pythagorean minor 6th and minor 3rd approximate the points of division between their respective base intervals, they are disturbed almost as much by their adjuncts as by their interpretations. The interpretation of the minor 6th is less discernible and therefore less disturbing than that of the minor 3rd, but the adjunct of the minor 6th is much more disturbing than that of the minor 3rd. Furthermore, the outside interval of the sixth, being conjoint to both base intervals, must be the more disturbing. It should not cause wonder, then, that the Pythagorean minor 6th sounds more dissonant than the Pythagorean minor 3rd.

The equally tempered minor 6th and minor 3rd, having deviations less than 20 cents in magnitude, are more consonant than the corresponding Pythagorean intervals; but they are still not acceptable and only moderately recognizable as tunings of their interpretations. Consequently, they are disturbed more by their interpretations than by their adjuncts. This tends to make the minor third the more dissonant because of the greater discernibleness of its interpretation, but the adjuncts and the outside intervals disturb the minor 6th more and thus ensure that it remains more dissonant than the minor third.

Let it be given that two indiscernible intervals have equally discernible interpretations, and let them deviate from their interpretations by different amounts. Then, if one deviates by at least 20 cents and the other by more than 30 cents, the one for which the deviation is <u>less</u> is the more disturbed. If one deviates by less than 20 cents and the other by no more than 30 cents, the one for which the deviation is <u>greater</u> is the more disturbed. This rule applies also to negative deviations if only their magnitudes are considered. The equally tempered diminished 5th deviates more from its interpretation than the minor 6th does; but both intervals deviate by less than 20 cents; therefore, if they had equally discernible interpretations, the dim.

5th would be the more disturbed by reason of its greater deviation. However, the dim. 5th has a more discernible interpretation and, because of this, is disturbed still more than the minor 6th is by its interpretation. Both intervals have the same adjunct, and their outside intervals are about equally disturbing. Therefore, it is not hard to see why the dim. 5th is considered to be the more dissonant.

If x/y is a point of division, it is not an acceptable tuning of either of the two discernible intervals conjoint to it (and to each other). Both give rise to the same rate of beating, and the deviation of x/y from the more discernible conjoint interval is the greater. Therefore, the disturbance due to one differs little, if any, from that due to the other. The Pythagorean minor 7th exemplifies this and provides an interesting comparison with the P minor 3rd. Its deviation from 7/4 is 27.3 cents and from 9/5 is -21.5 cents, exactly the same as the deviations of the P minor 3rd from its base intervals. Furthermore, 7/4 and 6/5 are equally discernible, and 9/5 and 7/6 are almost equally discernible. Taking these factors into consideration and ignoring the influence of the outside intervals would lead one to conclude that these two intervals have virtually the same degree of dissonance. The outside intervals 5/3 and 5/4 have essentially the same minor influence on both intervals, but the octave 2/1, being highly discernible and conjoint to both base intervals, lends a decisive increase to the dissonance of the minor 7th in spite of a difference in size of 204 cents. Because of this, the minor 7th has always been judged to be more dissonant than the minor 3rd.

If x/y <u>is</u> an acceptable tuning of a discernible interval, let this interval be the apex $x°/y°$ of a nucleus, let x'/y' be the largest discernible interval smaller than $x°/y°$, and let x''/y'' be the smallest discernible interval larger than $x°/y°$. Then x'/y' and x''/y'' are conjoint to $x°/y°$ and, being so, are the base intervals of the nucleus and potential sources of dissonance in x/y. On the other hand, it is a precondition to acceptance as a tuning that x/y differ from $x°/y°$ so little, if any, as to be disturbed little, if at all, by $x°/y°$. Because of this, our study of the dissonance of acceptable tunings of discernible intervals will not be concerned with differences between x/y and $x°/y°$, altho it is recognized that they exist.

That there is a special need for the base intervals to be conjoint to the apex can be shown as follows. If x/y is an exact tuning of $x°/y°$,

$$x = gx° \text{ and } y = gy° \qquad \text{K1}$$

where g is not only the fundamental frequency of the third tone but also the common difference between successive frequencies of the spectrum. Any beats heard in x/y, then, must have a fundamental rate of beating equal to g. In Section E, g'' is identified as the rate of beating that is heard when x/y approximates x'/y'. Supposing that x'/y' is a proximate discernible interval that disturbs x/y, we substitute from K1 into E3 with the result

$$\begin{aligned} g'' &= g(y'x° - x'y°) \\ &= gi'' \end{aligned} \qquad \text{K2}$$

and it is seen that g'' is the fundamental rate of beating only when $i'' = 1$. If x''/y'' is thought to disturb x/y, we substitute from K1 into E4 with the result

$$\begin{aligned} g' &= g(y°x'' - x°y'') \\ &= gi' \end{aligned} \qquad \text{K3}$$

and it is seen that g' is the fundamental rate of beating only when $i' = 1$. In either case, then, the beats arising from the base intervals have the fundamental rate only when they are conjoint to $x°/y°$.

Any outside intervals must also be conjoint to the apex for the same reason. Therefore, the outside interval smaller than x'/y' is $(x' - x°)/(y' - y°)$, the one larger than x''/y'' is $(x'' - x°)/(y'' - y°)$, and there will be an outside interval only when the apex is more discernible than the intervening base interval. Two apparent consequences (or principles) follow from the fact that these proximate discernible intervals are conjoint to $x°/y°$ and therefore give rise to one same rate of beating. First, they contribute equally or almost equally to the dissonance of $x°/y°$. Second, the disturbance from two or more of these intervals appears not to be cumulative, so that the dissonance induced by all is the same as that rising from any one as if it were the only one. While lacking direct proof of these two principles, the author is able to get consistent results in using them to determine which of two discernible intervals is the more dissonant. These results are displayed in Table IX, where the intervals $x°/y°$ are given in what is intended to be the order of increasing dissonance.

No proximate discernible intervals are given for the unison (1/1) because their number is limited by setting N equal to 7 and by requiring them to differ from $x°/y°$ no more than 231 cents (a frequency ratio of 8/7). Thus no dissonance is ascribed to the unison. In comparing the dissonance (or consonance) of the octave (2/1) with that of the twelfth (3/1), we note that the difference (182.4 cents) between 9/5 and 2/1 is the same as that between 3/1 and 10/3. However 10/3 is more discernible than 9/5 and therefore disturbs 3/1 more than 9/5 disturbs 2/1. Likewise 8/3 disturbs 3/1 more than 9/4 disturbs 2/1. These two findings are consistent and indicate that 3/1 is more dissonant (or less consonant) than 2/1. The difference (203.9 cents) between

Table IX

Discernible Intervals and the Proximate Discernible Intervals that Disturb Them

The intervals x°/y° listed here are discernible with N = 7 and are less than or equal to two octaves in size. Any interval that differs from x°/y° by more than 231 cents is not regarded as a disturbance. Here Q' = (y'x°)/(x'y°), and Q" = (y°x")/(x°y").

$\frac{x' - x^\circ}{y' - y^\circ}$	$\frac{x'}{y'}$	M log Q'	$\frac{x^\circ}{y^\circ}$	M log Q"	$\frac{x''}{y''}$	$\frac{x'' - x^\circ}{y'' - y^\circ}$
			1/1			
7/4	9/5	182.4	2/1	203.9	9/4	
	8/3	203.9	3/1	182.4	10/3	
7/2	11/3	150.6	4/1	203.9	9/2	
4/3	7/5	119.4	3/2	111.7	8/5	5/3
	7/3	119.4	5/2	111.7	8/3	
	10/3	84.5	7/2	80.5	11/3	4/1
	5/4	111.7	4/3	84.5	7/5	3/2
3/2	8/5	70.7	5/3	84.5	7/4	
	9/4	63.0	7/3	119.4	5/2	
	5/2	111.7	8/3	203.9	3/1	
	6/5	70.7	5/4	111.7	4/3	
	3/1	182.4	10/3	84.5	7/2	
	7/2	80.5	11/3	150.6	4/1	
	5/3	84.5	7/4	48.8	9/5	2/1
	7/6	48.8	6/5	70.7	5/4	
	2/1	203.9	9/4	63.0	7/3	
	4/3	84.5	7/5	119.4	3/2	
	3/2	111.7	8/5	70.7	5/3	
			7/6	48.8	6/5	
	7/4	48.8	9/5	182.4	2/1	

8/3 and 3/1 is the same as that between 4/1 and 9/2. Furthermore, 8/3 and 9/2 are equally discernible. This indicates that 3/1 and 4/1 must be equally dissonant and that 4/1, like 3/1, must be more dissonant than 2/1. Also, a direct comparison of 4/1 with 2/1 confirms this latter conclusion.

To compare 3/2 with 4/1, we note that 7/5 and 8/5 are more discernible that 11/3 and differ less from 3/2 than

11/3 does from 4/1, that 5/3 is more discernible than 7/2 and differs less from 3/2 than 7/2 does from 4/1, and that 4/3 is more discernible than 9/2 and differs exactly the same from 3/2 as 9/2 does from 4/1. In all these comparisons, the differences are such as to indicate beyond doubt that 3/2 is more dissonant than 4/1. It is easily seen that 5/2 and 7/2 are more dissonant than 3/2, but direct comparison of 7/2 with 5/2 offers no clear answer. However, 4/3 is definitely more dissonant than both 5/2 and 7/2. Comparisons like these thus establish largely the order of increasing dissonance of the intervals $x°/y°$. It is to be noted that this is also roughly but not strictly the order of decreasing discernibleness (or increasing obscurity), increases in dissonance usually being marked by increases in $x° + y°$.

The Pythagorean maj. 3rd and maj. 6th and the just large 4th and small 5th are listed in the order of increasing dissonance in Table VIII. The interpretations of these intervals are listed in the opposite order in Table IX, which is also judged to be the order of increasing dissonance. Such can be the effect of tuning or mistuning. When reference is made to the consonance or dissonance of a familiar consonant interval without specifying how it is tuned, it is perhaps best to assume that it is at least an acceptable tuning of its interpretation.

The consonance or dissonance of an interval has been an object of considerable interest from antiquity. In order to do justice to the subject, then, the author offers the following brief history. According to the Pythagorean theory of consonance, "the simpler the ratio of the two parts into which the vibrating string is divided, the more perfect is the consonance of the two sounds."[8] This simple observation may have had the effect of a definition of the term, but its application has not run a smooth course. Aristoxenus (4th century B.C.), who took issue with the followers of Pythagoras, preferred to make absolute distinctions, calling the unison, octave, fifth, and fourth consonances (or concords) and the other intervals dissonances (or discords), except that the expansion of consonant intervals by octaves did not alter their condition of consonance.[9] This classification nevertheless respected the Pythagorean theory in that the "consonant" intervals have simpler ratios than the other intervals. This opinion regarding consonance and dissonance was also held by the western Europeans during the early stages of organum (9th, 10th, and 11th centuries A.D.), when they too considered the unison, octave, fifth, and fourth to be consonant and the other intervals to be dissonant.

In the twelfth century, the perfect fourth came to be regarded by some as a dissonance. Franco of Cologne (13th century) once again classified the fourth as a consonance but at the same time accepted the major and minor thirds as consonances.[10] His complete classification is as follows:

perfect consonances	--	unison octave
medial consonances	--	fifth fourth
imperfect consonances	--	major third minor third
imperfect dissonances	--	major sixth minor seventh major second

---

8. Dayton C. Miller, Anecdotal History of the Science of Sound, p. 3. Pythagoras lived in the 6th century B.C.

9. The Harmonics of Aristoxenus, ed. and tr. by H.S. Macran.

10. Magistri Franconis, Ars Cantus Mensurabilis, Cap. XI, in Scriptorum de Musica Medii Aevi, ed. by E. de Coussemaker, vol. I, p. 129. See also Strunk, Source Readings in Music History, pp. 152-153.

perfect dissonances -- minor sixth
tritone
minor second
major seventh

Here we see a closer approach to the Pythagorean idea in that different degrees of consonance and dissonance are discerned and that in general the intervals with simpler ratios are considered more consonant.

Philippe de Vitri and Jean de Muris (14th century) elevated the fifth to the rank of perfect consonance, regarded the fourth as a dissonance, and included two more intervals, the major and minor sixths, among the imperfect consonances.[11] Soderlund,[12] in describing the sixteenth century style, gives essentially the same classification; and Fux[13] (18th century) gives exactly the same:

perfect consonances	--	unison
		octave
		fifth
imperfect consonances	--	major third
		minor third
		major sixth
		minor sixth
dissonances	--	major second
		minor second
		perfect fourth
		tritone
		major seventh
		minor seventh

According to Piston[14] this classification was observed in

---

11. Philippum de Vitriaco, _Ars Contrapunctus_, p. 27, Johannem de Muris, _Ars Contrapuncti_, p. 60, and _Ars Discantus_, p. 70, in _Scriptorum de Musica Medii Aevi_, ed. by E. de Coussemaker, vol. III.

12. _Direct Approach to Counterpoint_, p. 23.

13. _Steps to Parnassus_, ed. and tr. by Alfred Mann.

14. _Harmony_, p. 6.

the common practice of the nineteenth century also, giving it five centuries of recognition.

Helmholtz (late nineteenth century) regarded the fourth as a perfect consonance and the major sixth and major third as medial consonances.[15] Stumpf (same period), using a psychological approach and the term fusion instead of consonance, and determining the rank of the intervals by the vibration ratios of the tones, distinguished five grades of fusion, as follows:[16]

first grade	-- octave
second grade	-- fifth
third grade	-- fourth
fourth grade	-- thirds and sixths
fifth grade	-- the other intervals

Various psychologists have made studies of consonance in which intervals within the range of an octave were compared by a jury of observers for the purpose of arriving at their relative degrees of consonance. Malmberg, for example, using the terms smoothness, purity, and blending instead of consonance, achieved the following "order of merit of interval in the consonance-dissonance series":[17]

1. Octave
2. Fifth
3. Major sixth
4. Major third
5. Fourth
6. Minor sixth
7. Minor third
8. Diminished fifth
9. Minor seventh
10. Major second
11. Major seventh
12. Minor second

A reluctance to recognize the major sixth as a medial consonance is evident in the historical account; but most enigmatic is the perfect fourth, which was considered a consonance until the twelfth century and then became a source

15. Sensations of Tone, p. 194.

16. As quoted by Schoen, The Psychology of Music, pp. 48-49.

17. As quoted by Seashore, Psychology of Music, p. 132.

of divided opinion which persists even today. Piston points out that it was regarded as consonant when another tone was below its lower tone, which appears to weaken its character as a dissonance.[18] This confusion in regard to an interval so closely related to the perfect fifth, regarding which there has been no marked disagreement, must come from some faulty basic assumption; and that assumption is, in the author's opinion, that expanding an interval by one or more octaves does not change its degree of consonance or dissonance. This assumption dates back at least to Aristoxenus, and he made it clear in his Elements of Harmony that he did not care for fine distinctions in regard to this subject.

Everyone has been willing to distinguish the consonance of an interval from that of its octave inversion, but few besides Helmholtz have been willing to distinguish the consonance of an interval from that of its octave expansion. Yet the process of inverting an interval smaller than an octave differs from the process of increasing its size by an octave only in regard to which of its tones is raised or lowered an octave. Raising its upper tone an octave might conceivably produce just as much change in consonance as raising its lower tone. Starting with the twelfth, whose ratio (3/1) shows it to be the most consonant of the so-called "fifths," we can, by successively raising the original lower tone by octaves, pass from undisputed ground into controversial territory. The first raising produces the fifth (3/2); the second raising, the fourth (4/3); the third raisng, the eleventh (8/3); and so forth. The process of

18. The regular association of dissonance with "nonharmonic tones" from the 15th century thru the 19th has, in the author's estimation, caused many musicians to think of dissonance as a function rather than a quality. This could explain the dual classification, since the fourth admits of both "harmonic" and "nonharmonic" functions.

inverting the fifth to a fourth now appears merely as <u>part</u> of a larger process, and the series above appears to the author as gradually increasing in dissonance and not as suddenly changing from consonance to dissonance at the point of inversion.

It is also obvious from this series that the difference in degree of consonance between the 11th and 12th is much greater than that between the 4th and 5th. Therefore, it is to be expected, since no distinction was made between an interval and its octave expansion, and since much distinction was made between an interval and its inversion, that the degree of dissonance proper to the 11th came to be attributed at least in part to the 4th. According to our findings in Table IX, the 11th (8/3) is definitely more dissonant than the 4th (4/3); therefore, to attribute the dissonance of the 11th to the 4th would be to exaggerate the dissonance of the 4th by an appreciable amount. Likewise, the 11th is more dissonant than either the major 10th or the 12th; and, by analogy, the 4th could have come into a usage that treated it as being more dissonant than either the major third or the fifth. In the 14th century, the consistent use of intervals as large as the 11th and 12th in oblique and contrary motion began to appear; therefore, it is logical that from that time, the fourth being considered the same in musical practice as the eleventh, and the fifth the same as the twelfth, a greater difference in consonance has been attributed to the fourth and fifth than a direct comparison would warrant.

BIBLIOGRAPHY

Apel, Willi, Harvard Dictionary of Music. Cambridge: Harvard University Press, 1945.

Aristoxenus, The Harmonics of Aristoxenus. Ed. and tr. by Henry S. Macran. Oxford: The Clarendon Press, 1902.

Barbour, J. Murray, Tuning and Temperament, A Historical Study. East Lansing: Michigan State College Press, 2nd ed., 1953.

Corso, John F., and Lewis, Don, "Preferred Rate and Extent of the Frequency Vibrato." Journal of Applied Psychology. Vol 34 (1950), pp. 206-212.

De Coussemaker, Edmond (ed.), Scriptorum de Musica Medii Aevi. 3 vols. Paris: A. Durand, 1864. Reprint at Milan: Bollettino Bibliografico Musicale, 1931.

Fletcher, Harvey, Speech and Hearing. New York: D. Van Nostrand Co., 1929.

Fletcher, Harvey, Speech and Hearing in Communication. New York: D. Van Nostrand Co., 1953.

Fletcher, Harvey, and Munson, W.A., "Loudness, Its Definition, Measurement and Calculation." The Journal of the Acoustical Society of America. Vol. 5 (1933), pp. 82-108.

Fux, Johann J., Steps to Parnassus. Tr. and ed. by Alfred Mann. New York: W.W. Norton, 1943. First publ. 1725.

Galilei, Galileo, Dialogues Concerning Two New Sciences. Tr. from the Italian and Latin into English by H. Crew and A. de Salvio. New York: The Macmillan Co., 1914.

Helmholtz, Hermann L.F., On the Sensations of Tone as a Physiological Basis for the Theory of Music. Tr. by A.J. Ellis. London: Longmans, Green, and Co., 1930. First German edition, 1862; fourth German edition, 1877.

Hindemith, Paul, The Craft of Musical Composition. Tr. by A. Mendel. New York: Associated Music Publishers, 1945.

Lewis, Don, "Pitch: Its Definition and Physical Determinants." University of Iowa Studies in the Psychology of Music. Vol. 4 (1937), pp. 346-373.

Miller, Dayton Clarence, Anecdotal History of the Science of Sound. New York: The Macmillan Co. 1935.

Miller, Dayton Clarence, The Science of Musical Sounds. New York: The Macmillan Co., 1926.

Moe, Chesney R., "An Experimental Study of Subjective Tones Produced Within the Human Ear." The Journal of the Acoustical Society of America. Vol. 14 (1942), pp. 159-166.

Ohm, Georg S., "Ueber die Definition des Tones, nebst daran geknüpfter Theorie der Sirene und ähnlicher tonbildender Vorrichtungen." Annalen der Physik und Chemie. Ser. 2, vol. 59 (1843), pp. 513-565.

Partch, Harry, Genesis of a Music. Madison: The University of Wisconsin Press, 1949.

Piston, Walter, Harmony. New York: W.W. Norton, 1941.

Plomp, R., "Detectability Threshold for Combination Tones." The Journal of the Acoustical Society of America. Vol. 37, No. 6 (June 1965), pp. 1110-1123.

Pole, William, The Philosophy of Music. New York: Harcourt, Brace & Co., 1924.

Pratt, Carroll C., "Tonal Fusion." Psychological Review. Vol. 41 (1934), p. 95.

Roederer, Juan G., Introduction to the Physics and Psychophysics of Music, 2nd ed. New York: Springer-Verlag, 1975. & 1979

Sachs, Curt, Our Musical Heritage. New York: Prentice-Hall, 1948.

Saunders, F.A., "Analyses of the Tones of a Few Wind Instruments." The Journal of the Acoustical Society of America. Vol. 18 (1946), pp. 395-401.

Saunders, F.A., "The Mechanical Action of Violins." The Journal of the Acoustical Society of America. Vol. 9 (1937), pp. 81-98.

Schoen, Max, "An Experimental Study of the Pitch Factor in Artistic Singing." Psychol. Monogr. Vol. 31 (1922), pp. 230-259.

Schoen, Max, The Psychology of Music. New York: The Ronald Press Co., 1940.

Seashore, Carl E., _Psychology of Music_. New York: McGraw-Hill Book Co., 1938.

Seashore, Carl E. (ed.), _University of Iowa Studies in the Psychology of Music_. Vol. 1 (The Vibrato), Iowa City: The State University of Iowa, 1932.

Shirlaw, Matthew, _The Theory of Harmony; An Inquiry into the Natural Principles of Harmony, With an Examination of the Chief Systems of Harmony from Rameau to the Present Day_. London: Novello & Co., after 1903 (1918?).

Shower, E.G., and Biddulph, R., "Differential Pitch Sensitivity of the Ear." _The Journal of the Acoustical Society of America_. Vol. 3 (1931), pp. 275-287.

Soderlund, Gustave F., _Direct Approach to Counterpoint in 16th Century Style_. New York: F.S. Crofts & Co., 1947.

Stevens, Stanley S., and Davis, Hallowell, _Hearing, Its Psychology and Physiology_. New York: John Wiley & Sons, 1938.

Stout, Barrett, "The Harmonic Structure of Vowels in Singing in Relation toPitch and Intensity." _The Journal of the Acoustical Society of America_. Vol. 10, no. 2 (1938), pp. 137-146.

Tartini, Giuseppe, _Trattato di musica secondo la vera scienza dell' armonia_. Padua, 1754. A facsimlie of the 1754 Padua edition, New York: Broude Brothers, 1966.

Tobias, Jerry V. (ed.), _Foundations of Modern Auditory Theory_, Vol. I. New York: Academic Press, 1970.

Wever, Ernest Glen, _Theory of Hearing_. New York: John Wiley and Sons, 1949.

INDEX OF SYMBOLS

The chapter, section, and equation number where it is first defined or introduced are given after each symbol.

A	4C16	An interpolation between x' + y' and x" + y".
Arc	4G2	An angle relating x/y to x'/y', x"/y", and their point of division.
a,b	2F	Degree of a primary tone or an aural harmonic.
a	4H	Greatest common divisor of x' and x".
b	4H	Greatest common divisor of y' and y".
c	1F	Speed of sound in air.
cps	1A	Cycles per second.
D	1F	Density of air at atmospheric pressure.
d	1F	Differential of a variable.
db	1F	Decibels.
F	4J16	Center frequency of two beating tones.
f	2F	Frequency of a tone in the aural spectrum.
f°	3A8	Frequency number of the tone of frequency f.
f'	3C1	Frequency of another spectral tone.
f'°	3C2	Frequency number of the tone of frequency f'.
G	3B1	Loudness index of mx + ny.
G'	3D1	Loudness index of m'x + n'y.
g	3A	Greatest common divisor of x and y when they are commensurable.
g'	4E4	The rate of beating when x/y approximates x"/y".
g"	4E3	The rate of beating when x/y approximates x'/y'.
H	2A24	Average intensity of an intertone.
I	1F6	Sound intensity. Work done per second.
I_o	1F	Reference intensity.
IL	1F11	Intensity level.
i	4A1	A determinant relating x"/y" to x'/y'.
i'	4A2	A determinant relating x"/y" to x°/y°.
i"	4A3	A determinant relating x°/y° to x'/y'.

k	3C	The greatest common divisor of m - m' and n' - n when the tones coincide.
l_r	1G	Integer or integral variable.
ln	1B	Natural logarithm -- log to the base 2.718282...
M	4F11	1200/log 2 -- the modulus for conversion to cents. See also Ch. 1, Sec. B.
m	2E	Coefficient of x. Integer.
m'	3C	Another coefficient of x.
N	2F	Number of audible partials in a typical musical tone.
	3F	Maximum index of an audibly significant spectral tone.
n	1G	Number of partials in a complex tone.
	2A	Number of tones in a spectrum.
	2E	Coefficient of y. Integer.
n'	3C	Another coefficient of y.
P	2A	Phase angle of the intertone.
P_r	1G6	Phase angle of partial r.
	2A	Phase angle of tone r.
P_s	1G14	Phase angle of partial s.
p	1F	Excess air pressure due to sound.
Q	4I	The interval by which x/y deviates from its interpretation.
q_r	2A	Phase displacement.
R	1B	Frequency ratio of a unit interval.
	4J10	Frequency ratio of the tones in the highest primary adjacency.
r,s	1G	1, 2, 3, ... n. Integers denoting partials of a complex tone.
	2A	1, 2, 3, ... n. Integers denoting tones of a spectrum.
S	2A25	Salient frequency of an intertone.
s	1B	Size of an interval.
	1G	(See r,s)
T_r	1E	Time when maximum velocity of the rth partial or tone is first reached.

t	1E	Time.
U_r	1E	Displacement amplitude of a partial tone.
$U_\circ$	1E	A static displacement of air particles due to sound.
u	1E	Displacement of air particles due to sound.
V	2A	Velocity amplitude of the intertone.
V_r	1E	Velocity amplitude of a partial tone.
	2A	Velocity amplitude of a tone of a spectrum.
v	1E	Velocity of air particles due to sound.
W	1F	Work done by excess air pressure on a unit area of particles.
$W_\circ$	1F5	Work done per cycle.
w	2A3	
X	3A	Complex tone of frequency x.
x	1B	Frequency in cycles per second of the higher tone of an interval.
x°	1D	x divided by the greatest common divisor of x and y.
x'	4A	The greater of a pair of relatively prime integers.
x''	4A	The greater of a pair of relatively prime integers.
Y	3A	Complex tone of frequency y.
y	1B	Frequency in cycles per second of the lower tone of an interval.
y°	1D	y divided by the greatest common divisor of x and y.
y'	4A	The lesser of a pair of relatively prime integers.
y''	4A	The lesser of a pair of relatively prime integers.
Δt	2B	Period of one pulsation or beat.
μ	3F8	$= 0$ when $x^\circ \pm y^\circ$ is even.
		$= 1$ when $x^\circ \pm y^\circ$ is odd.
π	1E	Ratio of the circumference of a circle to its diameter.
Σ	1G	Summation sign.
Ω	1E1	Frequency in radians per second.
$\cong$		Is approximately equal to.

INDEX OF FIGURES, TABLES, AND PLOTS

SUBJECT INDEX

The initial price of this book is $25. Because of inflation, the price must increase annually. An inflation rate of 4.7% per year gives us the following price schedule:

Year	Price
1990	$25.00
1991	26.20
1992	27.40
1993	28.70
1994	30.00